

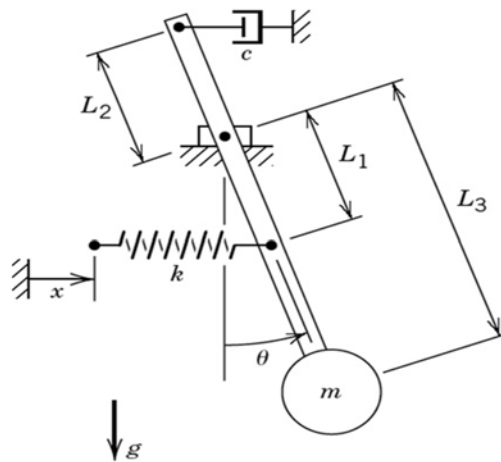
MCG4308 Midterm Exam February 2016

Question 1

The mass m shown in the figure is attached to a rigid lever which has negligible mass and negligible pivot friction.

The displacement x is a fixed, specified distance. When x and θ are zero, the spring is at its free length. Assume that θ is small.

- derive the equation of motion for θ .
- Find the equivalent mass and equivalent stiffness of this system
- what is the natural frequency of this system
- For what value of c will this system be critically damped?



Solution

Equation of motion can be found by taking moments about the pivot point. The moment of inertia of a point mass about the fixed point is given by $I_m = m \cdot L_3^2$.

- For small angles, we can write the sum of moments about the pivot as

$$I_m \cdot \ddot{\theta} = -m \cdot g \cdot L_3 \cdot \theta - k \cdot L_1 \cdot (x - L_1 \cdot \theta) - c \cdot L_2 \cdot \dot{\theta} \cdot L_2$$

Collecting terms, we get

$$I_m \cdot \ddot{\theta} + c \cdot L_2^2 \cdot \dot{\theta} + (m \cdot g \cdot L_3 + k \cdot L_1^2) \cdot \theta = k \cdot L_1 \cdot x$$

- From the equation, the equivalent stiffness is given by

$$k_{eq} = (m \cdot g \cdot L_3 + k \cdot L_1^2)$$

The equivalent mass is given by the moment of inertia of the mass $m_{eq} = m \cdot L_3^2$

c) Using the fact that $I_m = m \cdot L_3^2$ The natural frequency is given by

$$\omega_n = \sqrt{\frac{(m \cdot g \cdot L_3 + k \cdot L_1^2)}{m \cdot L_3^2}}$$

Via energy methods

Kinetic energy is given by

$$T = \frac{1}{2} m \cdot (L_3 \cdot \dot{\theta})^2$$

$$T = \frac{1}{2} m L_3^2 \left(\frac{d}{dt} \theta(t) \right)^2 \quad (1.1)$$

Therefore, equivalent mass is

$$m_{eq} = m \cdot L_3^2$$

$$m_{eq} = m L_3^2 \quad (1.2)$$

Potential energy is

$$V = \frac{1}{2} \cdot k \cdot (L_1 \cdot \theta - x)^2 + m \cdot g \cdot L_3 (1 - \cos(\theta))$$

$$\text{with small angles } V = \frac{1}{2} \cdot k \cdot (L_1 \cdot \theta - x)^2 + m \cdot g \cdot L_3 \left(\frac{1}{2} \theta^2 \right)$$

$$V = \frac{1}{2} k (\theta L_1 - x)^2 + m g L_3 \left(\frac{1}{2} \theta^2 \right) \quad (1.3)$$

Hence equivalent stiffness is

$$k_{eq} = k \cdot L_1^2 + m \cdot g \cdot L_3$$

$$k_{eq} = g m L_3 + k L_1^2 \quad (1.4)$$

Equation of motion via energy

Energy lost through the damper is Force produced by damper *velocity at damper so

$$E_{loss} = -c \cdot (L_2 \dot{\theta}) \cdot (L_2 \cdot \dot{\theta})$$

$$E_{loss} = -c L_2^2 \left(\frac{d}{dt} \theta(t) \right)^2 \quad (1.5)$$

Equation of motion comes from $\frac{d}{dt} (T + V) = E_{loss}$

$$m \cdot L_3^2 \cdot \dot{\theta} \cdot \ddot{\theta} + k (\theta L_1 - x) \cdot (L_1 \cdot \dot{\theta}) + m g L_3 \theta \cdot \dot{\theta} = -c \cdot (L_2 \dot{\theta}) \cdot (L_2 \cdot \dot{\theta})$$

Dividing through by $\dot{\theta}$ and cleaning up gives

$$m \cdot L_3^2 \cdot \ddot{\theta} + c \cdot L_2^2 \cdot \dot{\theta} + (m \cdot g \cdot L_3 + k \cdot L_1^2) \cdot \theta = k \cdot L_1 \cdot x$$

$$m L_3^2 \left(\frac{d^2}{dt^2} \theta(t) \right) + c L_2^2 \left(\frac{d}{dt} \theta(t) \right) + (g m L_3 + k L_1^2) \theta = k L_1 x \quad (1.6)$$

d) System is critically damped when coefficient of $\dot{\theta}$ (c_{eq} = equivalent c') is equal to $2\sqrt{m_{eq} \cdot k_{eq}}$

$$c \cdot L_2^2 = 2 \cdot \sqrt{m_{eq} \cdot k_{eq}}$$

$$c L_2^2 = 2 \sqrt{m_{eq} k_{eq}} \quad (1.7)$$

$$c \cdot L_2^2 = 2 \cdot \sqrt{(m \cdot g \cdot L_3 + k \cdot L_1^2) \cdot m \cdot L_3^2}$$

$$c L_2^2 = 2 \sqrt{(g m L_3 + k L_1^2) m L_3^2} \quad (1.8)$$

Question 2

A precision grinding machine is supported on an isolator consisting of two identical springs and a damper. The floor, on which the machine is mounted, is subjected to a harmonic disturbance due to the operation of an unbalanced pump that is located nearby.

a) Write the equation of motion for this system.

b) Since the motion of the base is harmonic, $y(t) = Y \cdot \sin(\omega \cdot t)$, calculate the magnitude and phase of the steady state response of the system.

c) What is the displacement transmissibility of the base motion?

d) It is discovered that the operating frequency of the unbalanced pump is close to the natural frequency of the grinding machine so that the amplitude of the motion of the grinding machine is too high. Which of the following techniques can be used, with confidence, to reduce the motion of the machine? Choose ALL that apply and explain.

A. Reduce the mass of the machine

B. Increase the mass of the machine

C. Reduce the stiffness of the isolator (springs)

D. Increase the stiffness of the isolator (springs)

E. Reduce the damping of the isolator

F. Increase the damping of the isolator

Equation: $m\ddot{x} + c\dot{x} + 2kx = c\dot{y} + 2ky$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\dot{y} + \omega_n^2y \quad \omega_n^2 = \frac{2k}{m}$$

Sub into equation: $y(t) = Ye^{i\omega t}$ $x(t) = Xe^{i(\omega t + \phi)} = Xe^{i\phi}e^{i\omega t}$

$$[-\omega^2 + 2\zeta\omega_n\omega i + \omega_n^2]Xe^{i\phi}e^{i\omega t} = [2\zeta\omega_n\omega i + \omega_n^2]Ye^{i\omega t}$$

Divide both sides by ω_n^2 and $e^{i\omega t}$, remember $r = \omega/\omega_n$

$$[-r^2 + 2\zeta r i + 1]Xe^{i\phi} = [2\zeta r i + 1]Y$$

$$Xe^{i\phi} = \frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i} Y$$

So match up magnitudes and phases:

$$X = \text{magnitude of } \left[\frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i} \right] Y = Y \sqrt{\frac{(2\zeta r)^2 + 1}{(1 - r^2)^2 + (2\zeta r)^2}}$$

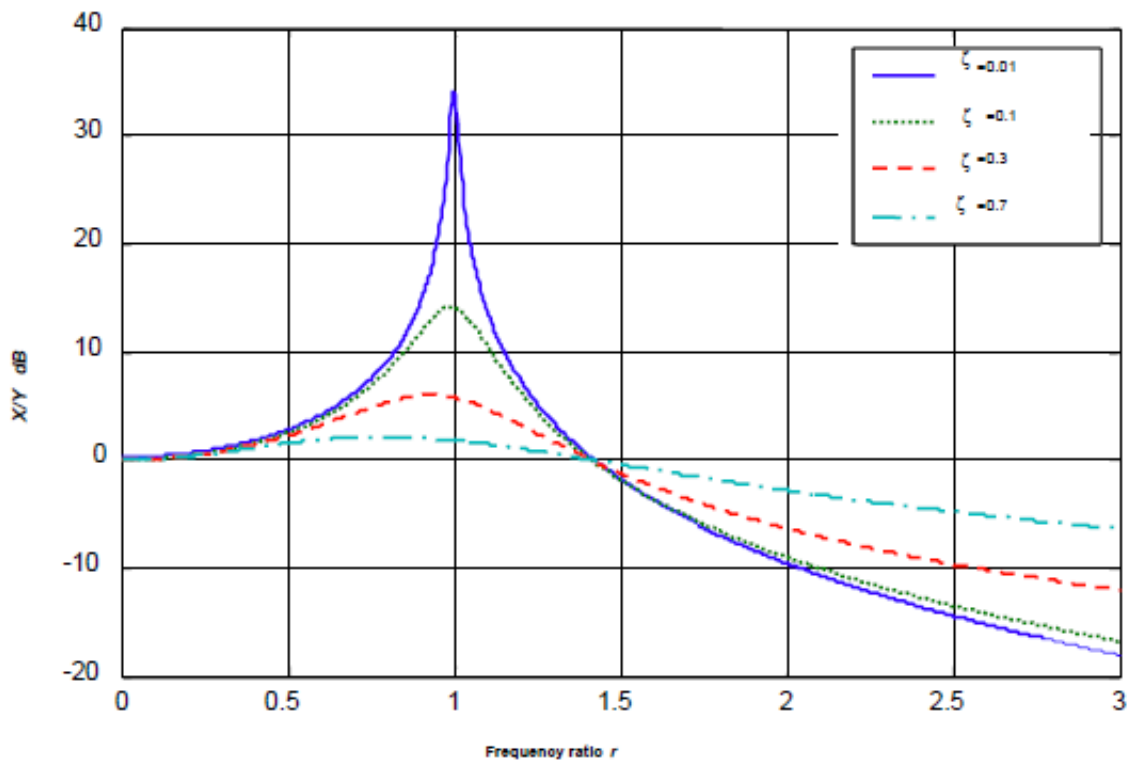
$$\phi = \text{phase of } \left[\frac{2\zeta r i + 1}{1 - r^2 + 2\zeta r i} \right] = \arctan\left(\frac{2\zeta r}{1}\right) - \arctan\left(\frac{2\zeta r}{1 - r^2}\right)$$

so the system response is $x(t) = X \sin(\omega t + \phi)$

c) The transmissibility ratio is given by

$$\frac{X}{Y} = \sqrt{\frac{(2\zeta r)^2 + 1}{(1 - r^2)^2 + (2\zeta r)^2}}$$

which is just the magnitude part of part (b) rearranged as a ratio of input response magnitude to input magnitude.



d) To decrease amplitude of response, need to increase damping or change the natural frequency of the system. Changing either the mass or stiffness will change the natural frequency of the system. **So all situations except for (E) apply.** We can change the natural frequency (change m or k) or increase the damping.

Question 3

The motion of a linear spring-mass damper-system is described by the following equation

$$ode := 1 \cdot \frac{d^2}{dt^2} x(t) + 2 \cdot \frac{d}{dt} x(t) + 2 \cdot x(t) = 20 + \exp(-t) :$$

You may consider that that system is initially at rest (zero initial conditions).

- Find and sketch** the form of the free vibration (homogeneous) response.
- Is the system underdamped, overdamped or critically damped?
- Find and sketch **the total response** of the system.
- Identify which portions of the total response are the transient and steady state responses.

(a) Homogeneous solution/Free vibration Response

The transient solution is the solution to

$$ode2 := 1 \cdot \frac{d^2}{dt^2} x_h(t) + 2 \cdot \frac{d}{dt} x_h(t) + 2 \cdot x_h(t) = 0$$

$$\frac{d^2}{dt^2} x_h(t) + 2 \left(\frac{d}{dt} x_h(t) \right) + 2 x_h(t) = 0 \quad (3.1)$$

For the transient/homogeneous solution, guess $x_h(t) = \exp(s \cdot t)$

$$x_h(t) = e^{st} \quad (3.2)$$

Substitute this into the equation of motion. $\text{subs}(x_h(t) = \exp(s \cdot t), \text{ode2})$

$$\frac{\partial^2}{\partial t^2} e^{st} + 2 \left(\frac{\partial}{\partial t} e^{st} \right) + 2 e^{st} = 0 \quad (3.3)$$

Simplify and divide through by the exponential to get the characteristic polynomial

$$\text{simplify} \left(\frac{(3.3)}{\exp(s \cdot t)} \right)$$

$$s^2 + 2s + 2 = 0 \quad (3.4)$$

Solve the characteristic polynomial $S := \text{solve}((3.4))$

$$-1 + I, -1 - I \quad (3.5)$$

COMPLEX CONJUGATE ROOTS WITH REAL PART: This gives us the *form* of the homogeneous solution:

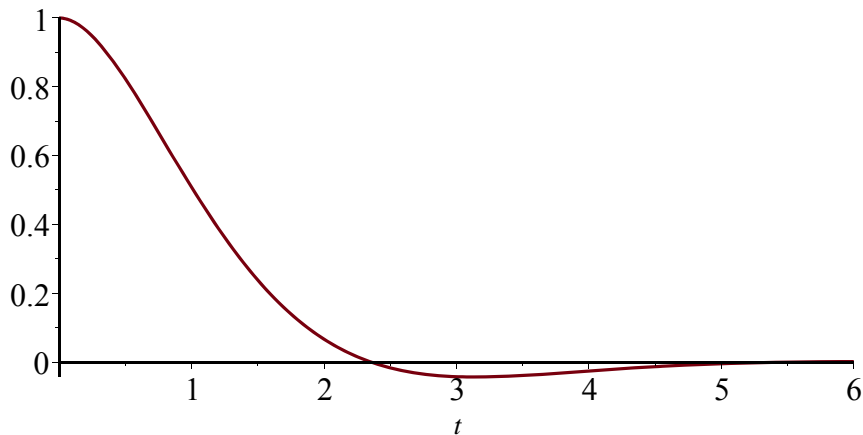
$$x_h(t) = \exp(-t) \cdot (a_1 \cdot \cos(t) + a_2 \cdot \sin(t))$$

$$x_h(t) = e^{-t} (a_1 \cos(t) + a_2 \sin(t)) \quad (3.6)$$

$\text{assign}((3.6))$

We won't find the values of the actual a_1 and a_2 until after we plug in the initial conditions into the TOTAL solution (homogeneous + particular(s)). But in the meantime, we can still get an idea of what the homogenous solutions looks like by picking some values for the a's and plotting.

$\text{plot}(\text{subs}(a_1 = 1, a_2 = 1, x_h(t)), t = 0 .. 6)$



(b) System is underdamped

(c) Total response

Particular Solution 1

Choose form for particular solution -- choose something that looks like the forcing function itself, according to the rough rules given in the table.

$$x_p(t) = B \cdot \exp(-t)$$

$$x_p(t) = B e^{-t} \quad (3.7)$$

Substitute this into the original differential equation with only the exponential forcing term on the RHS

$$\text{subs}\left(x(t) = \text{rhs}((3.7)), 1 \cdot \frac{d^2}{dt^2} x(t) + 2 \cdot \frac{d}{dt} x(t) + 2 \cdot x(t) = \exp(-t)\right)$$

$$\frac{\partial^2}{\partial t^2} (B e^{-t}) + 2 \left(\frac{\partial}{\partial t} (B e^{-t}) \right) + 2 B e^{-t} = e^{-t} \quad (3.8)$$

Rearrange by collecting and cleaning up:
`collect(simplify((3.8)), t)`

$$B e^{-t} = e^{-t} \quad (3.9)$$

Matching up the coefficients of powers of t on the left and right hand sides:
`coeff(lhs((3.9)), exp(-t)) = coeff(rhs((3.9)), exp(-t))`

$$B = 1 \quad (3.10)$$

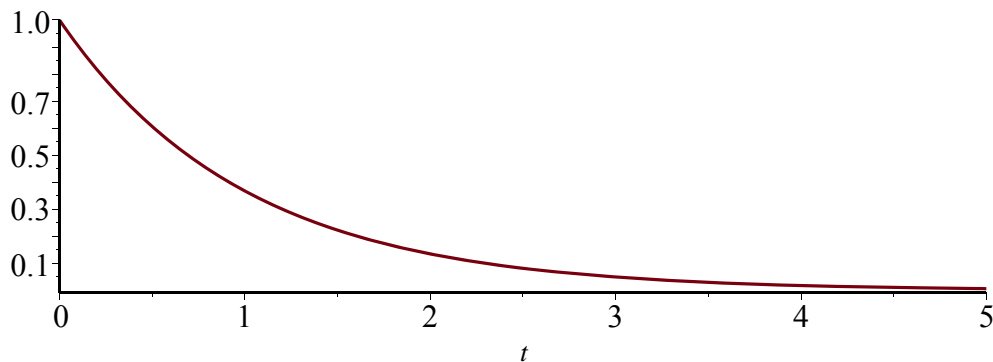
So particular solution 1 is

$$x_{p1}(t) = \exp(-t)$$

$$x_{p1}(t) = e^{-t} \quad (3.11)$$

`assign((3.11))`

Plot the particular solution `plot(x_{p1}(t), t=0..5)`



In this case, the particular solution dies off and in fact goes to zero as time goes on.

Particular solution 2

Choose form for particular solution -- choose something that looks like the forcing function itself, according to the rough rules given in the table.

$$x_{p2}(t) = A$$

$$x_{p2}(t) = A \quad (3.12)$$

Substitute this into the original differential equation with only the 20 on the Right Hand Side

$$\text{subs}\left(x(t) = \text{rhs}((3.12)), 1 \cdot \frac{d^2}{dt^2} x(t) + 2 \cdot \frac{d}{dt} x(t) + 2 \cdot x(t) = 20\right)$$

$$\frac{\partial^2}{\partial t^2} A + 2 \left(\frac{\partial}{\partial t} A \right) + 2 A = 20 \quad (3.13)$$

Rearrange by collecting and cleaning up:
`collect(simplify((3.13)), t)`

$$2 A = 20 \quad (3.14)$$

`solve((3.14), A)`

$$10 \quad (3.15)$$

So the second particular solution is given by

$$x_{p2}(t) = 10$$

$$x_{p2}(t) = 10 \quad (3.16)$$

assign((3.16))

Total Solution

$$x_{total}(t) = x_h(t) + x_{p1}(t) + x_{p2}(t)$$

$$x_{total}(t) = e^{-t} (a_1 \cos(t) + a_2 \sin(t)) + e^{-t} + 10 \quad (3.17)$$

assign((3.17))

THIS IS THE TOTAL RESPONSE (equation (3.17)). You do not actually need to find the unknowns a_1 and a_2 to find the *form* of the solution and thus characterize the response

However, *IF* you did, the correct procedure is to NOW plug in the initial conditions.

$$subs(t=0, x_{total}(t)) = rhs(IC[1])$$

$$e^0 (a_1 \cos(0) + a_2 \sin(0)) + e^0 + 10 = 0 \quad (3.18)$$

simplify((3.18))

$$a_1 + 11 = 0 \quad (3.19)$$

$$subs(t=0, \dot{x}_{total}(t)) = rhs(IC[2])$$

$$-e^0 (a_1 \cos(0) + a_2 \sin(0)) + e^0 (-a_1 \sin(0) + a_2 \cos(0)) - e^0 = 0 \quad (3.20)$$

simplify((3.20))

$$-a_1 + a_2 - 1 = 0 \quad (3.21)$$

solve({(3.19), (3.21)})

$$\{a_1 = -11, a_2 = -10\}$$

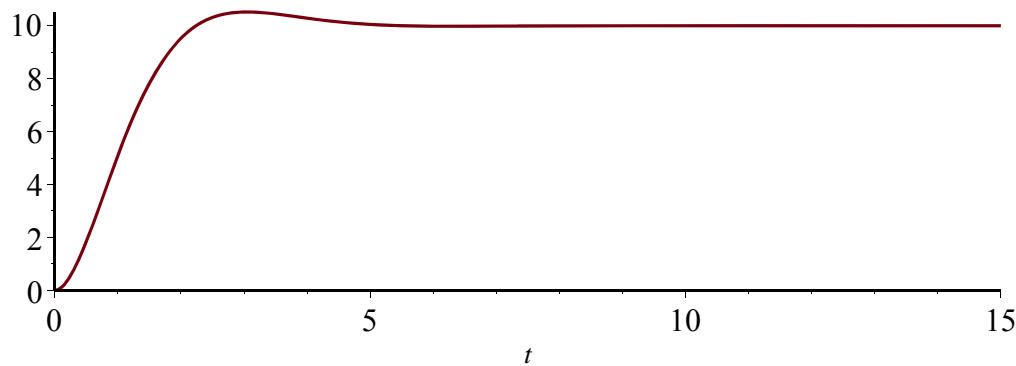
$$x(t) = subs((3.22), x_{total}(t))$$

$$x(t) = e^{-t} (-11 \cos(t) - 10 \sin(t)) + e^{-t} + 10 \quad (3.23)$$

assign((3.23))

Now plot the total solution

plot(x(t), t=0..15)



Here the homogeneous solution dies off (because of the negative real part in the exponential) and then the exploding particular solution takes over.

(d) The transient portion of the response is that part that dies off:

$$x_{transient}(t) = e^{-t} (-11 \cos(t) - 10 \sin(t)) + e^{-t}$$

$$x_{transient}(t) = e^{-t} (-11 \cos(t) - 10 \sin(t)) + e^{-t} \quad (3.24)$$

The steady state part of the response is the part that does not die off

$$x_{ss}(t) = 10$$

$$x_{ss}(t) = 10$$

(3.25)