MCG 4303 – Mechanical Vibration Analysis

Friday, February 17, 2006

Midterm Exam

Comments and general instructions:

1. PUT YOUR NAME AND STUDENT NUMBER ON YOUR EXAM BOOKLET!
2. There are ***five problems*** (100 points total), each weighted 20 points. Be sure to attempt all problems. There is a ***bonus problem*** worth 10 extra points.
3. Potentially useful mathematical formulas are on the last two pages.
4. Be neat and clear in your presentation, and show all work for full credit.
5. This exam is closed book.
6. Time: **90 minutes.**

Good luck!

1) **(20 points)** For the system shown in Figure 1:
   a) Determine the kinetic and potential energies of the system
   b) Determine the effective mass and stiffness of the system.
   c) Use Raleigh's energy method to determine the natural frequency of vibration.

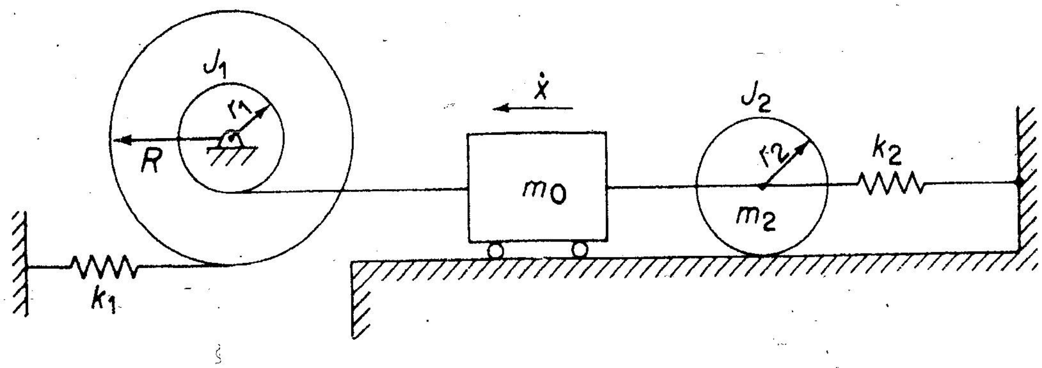


Figure 1

2) **(20 points)** A mass-spring-damper system with m=1 kg, k= 12 N/m and c = 7 Ns/m is given an initial displacement $x_0 = 0$ and initial velocity $v_0=5$ m/s.
   a) Determine the resulting motion
   b) Is the system overdamped, critically damped or underdamped?

3) **(20 points)** A mass-spring-damper system is subject to harmonic forcing of the form $F(t)=F_0e^{i\omega t}$. Use complex algebra to determine the amplitude and phase of the system response with respect to the input force.

4) **(20 points)** Find the response of an undamped spring-mass system with natural frequency ω_n under the forcing function $f(t) = e^{-t}$ and zero initial conditions.

5) **(20 points)** Consider the rectangular wave shown in Figure 2.
   a) Is this rectangular wave an odd function, an even function or neither?
   b) Determine the Fourier series for the rectangular wave shown in Figure 2.

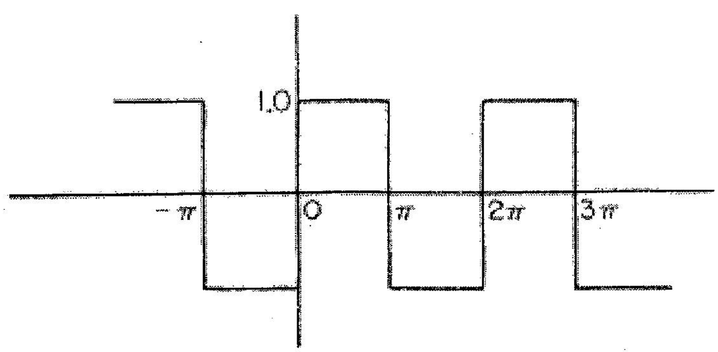


Figure 2

6) BONUS QUESTION: **(10 bonus points)** What is the Fourier series for cos(t)?

Useful formulas

Quadratic formula: roots of ax^2+bx+c are given by $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\int udv = uv - \int vdu$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i} = \mathrm{Im}\left\{e^{iu}\right\} \qquad \cos(u) = \frac{e^{iu} + e^{-iu}}{2} = \mathrm{Re}\left\{e^{iu}\right\}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)dt$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\sin(n\omega t)dt$$

Short Table of Laplace Transforms

$f(t)$	$F(s) = L[f(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{1}{s}e^{-as}$

Trigonometric Identities

Reciprocal

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$
$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$
$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Pythagorean

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Negative Angle

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Addition and Subtraction

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Periodicity

$$\sin(x+2\pi) = \sin x \qquad \cos(x+2\pi) = \cos x$$
$$\csc(x+2\pi) = \csc x \qquad \sec(x+2\pi) = \sec x$$
$$\tan(x+\pi) = \tan x \qquad \cot(x+\pi) = \cot x$$

Cofunction

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$
$$\tan x = \cot\left(\frac{\pi}{2} - x\right) \qquad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$
$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \qquad \csc x = \sec\left(\frac{\pi}{2} - x\right)$$

Product

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$
$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$
$$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$
$$\cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

Factoring

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Double Angle

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = 1 - 2\sin^2 x$$
$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$
$$\cos(2x) = 2\cos^2 x - 1$$

Half-Angle

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$$

① $KE = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_o \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$

$$\theta_1 = \frac{x}{r_1} \qquad \theta_2 = \frac{x}{r_2}$$

$$KE = \frac{1}{2}\left[J_1 \frac{\dot{x}^2}{r_1^2} + m_o \dot{x}^2 + m_2 \dot{x}^2 + J_2 \frac{\dot{x}^2}{r_2^2} \right]$$

$$= \frac{1}{2}\left[\frac{J_1}{r_1^2} + m_o + m_2 + \frac{J_2}{r_2^2} \right] \dot{x}^2$$

$$m_{eq} = \frac{J_1}{r_1^2} + m_o + m_2 + \frac{J_2}{r_2^2}$$

$$PE = \frac{1}{2} k_2 x^2 + \frac{1}{2} k_1 (R\theta_1)^2$$

$$= \frac{1}{2} k_2 x^2 + \frac{1}{2} k_1 \left(R \frac{x}{r_1}\right)^2$$

$$= \frac{1}{2}\left[k_2 + \frac{k_1 R^2}{r_1^2} \right] x^2$$

$$k_{eq} = k_2 + \frac{k_1 R^2}{r_1^2}$$

$$\omega_n^2 = \frac{k_{eq}}{m_{eq}} = \frac{k_2 + k_1 R^2 / r_1^2}{J_1/r_1^2 + m_o + m_2 + J_2/r_2^2}$$

(2) $\ddot{x} + 7\dot{x} + 12x = 0$ $\quad$ $x(0) = 0$

$\dot{x}(0) = 5$

$x = e^{\lambda t}$ $\Rightarrow$ characteristic equation

$$\lambda^2 + 7\lambda + 12 = 0$$

$$(\lambda+3)(\lambda+4) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = -4$$

$$\lambda = \frac{-7 \pm \sqrt{49 - 4\cdot(12)}}{2}$$

$$\lambda = \frac{-7 \pm 1}{2}$$

$$x(t) = C_1 e^{-3t} + C_2 e^{-4t}$$

$$\dot{x}(t) = -3C_1 e^{-3t} - 4C_2 e^{-4t}$$

$x(0) = C_1 + C_2 = 0 \quad \Rightarrow \quad C_1 = -C_2$

$\dot{x}(0) = -3C_1 - 4C_2 = -3C_1 + 4C_1 = C_1 = 5$

$C_1 = 5 \quad C_2 = -C_1 = -5$

$\Rightarrow \quad x(t) = 5e^{-3t} - 5e^{-4t}$

2 real roots $\lambda \Rightarrow$ overdamped system.

alternatively $c_c = 2m\omega_n = 2\sqrt{12} = 6.9$

$c = 7 > c_c$ $\Rightarrow$ overdamped.

(3) $m\ddot{x} + c\dot{x} + kx = Ye^{i\omega t}$

$x = Xe^{i\omega t}$ particular sol^n - steady state

$$m(-\omega^2)X\cancel{e^{i\omega t}} + ci\omega X\cancel{e^{i\omega t}} + kX\cancel{e^{i\omega t}} = Y\cancel{e^{i\omega t}}$$

$$\left[(k - m\omega^2) + ci\omega\right]X = Y$$

$$X = \frac{Y}{(k-m\omega^2) + ci\omega}$$

$$= \frac{Ye^{-i\phi}}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \qquad \phi = \arctan\left[\frac{c\omega}{(k-m\omega^2)}\right]$$

$$\Rightarrow \quad x = \frac{Y}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\, e^{i(\omega t - \phi)}$$

(4)

$$\ddot{x} + \omega_n^2 x = e^{-t} \qquad x(0) = 0$$
$$\dot{x}(0) = 0.$$

$$x = x_h + x_p$$

$$x_h = C_1 \cos\omega_n t + C_2 \sin\omega_n t$$

for x_p Guess $x_p = C_3 e^{-t}$

$$\Rightarrow \quad C_3 e^{-t} + \omega_n^2 C_3 e^{-t} = e^{-t}$$
$$\Rightarrow \quad (1+\omega_n^2) C_3 = 1$$
$$\Rightarrow \quad C_3 = \frac{1}{1+\omega_n^2}$$

$$x = C_1 \cos\omega_n t + C_2 \sin\omega_n t + \frac{e^{-t}}{1+\omega_n^2}.$$

$$\dot{x}(t) = -C_1 \omega_n \sin\omega_n t + C_2 \omega_n \cos\omega_n t - \frac{e^{-t}}{1+\omega_n^2}.$$

$$x(0) = C_1 + \frac{1}{1+\omega_n^2} = 0 \quad \Rightarrow \quad C_1 = \frac{-1}{1+\omega_n^2}$$

$$\dot{x}(0) = C_2 \omega_n - \frac{1}{1+\omega_n^2} = 0 \quad \Rightarrow \quad C_2 = \frac{1}{\omega_n(1+\omega_n^2)}$$

$$x(t) = \frac{-1}{1+\omega_n^2} \cos\omega_n t + \frac{1}{\omega_n(1+\omega_n^2)} \sin\omega_n t + \frac{e^{-t}}{1+\omega_n^2}$$

$$= \frac{1}{(1+\omega_n^2)} \left[\frac{\sin\omega_n t}{\omega_n} - \cos\omega_n t + e^{-t} \right]$$

(4)

$m\ddot{x}+kx = e^{-t}$ $\qquad x(0)=0 \quad \dot{x}(0)=0$

$x = x_h + x_p$

$x_h = C_1 \cos w_n t + C_2 \sin w_n t$

$x_p \Rightarrow$ Guess $x_p = C_3 e^{-t}$

$$(C_3 m + k C_3) e^{-t} = e^{-t} \Rightarrow C_3 = \frac{1}{m+k}$$

$$x = C_1 \cos w_n t + C_2 \sin w_n t + \frac{e^{-t}}{m+k}$$

$$\dot{x} = -C_1 w_n \sin w_n t + C_2 w_n \cos w_n t - \frac{e^{-t}}{m+k}$$

$$x(0) = C_1 + \frac{1}{m+k} \Rightarrow C_1 = -\frac{1}{m+k}$$

$$\dot{x}(0) = C_2 w_n - \frac{1}{m+k} \Rightarrow C_2 = \frac{1}{w_n(m+k)}$$

$$\Rightarrow x(t) = \frac{1}{m+k}\left[-\cos w_n t + \frac{\sin w_n t}{w_n} + e^{-t}\right]$$

(4) $\ddot{x} + \omega_n^2 x = e^{-t} \qquad x(0) = 0 \qquad \dot{x}(0) = 0$

<u>LAPLACE</u>

Laplace $\quad X(s)(s^2+\omega_n^2) = \dfrac{1}{s+1}$

$$\Rightarrow \quad X(s) = \frac{1}{(s-1)(s^2+\omega_n^2)}$$

$$\frac{1}{(s+1)(s^2+\omega_n^2)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+\omega_n^2)}$$

$$1 = A(s^2+\omega_n^2) + (s+1)(Bs+c)$$

$s^0 \qquad 1 = A\omega_n^2 + C$

$s^2 \qquad 0 = A + B \qquad\qquad A = -B$

$s^1 \qquad 0 = +B + C \quad \Rightarrow \quad C = -B \qquad A = +C$

$$1 = A\omega_n^2 + A = A(\omega_n^2+1)$$

$$A = \frac{1}{\omega_n^2+1} \qquad B = \frac{-1}{\omega_n^2+1} \qquad C = \frac{+1}{\omega_n^2+1}$$

$$X(s) = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+\omega_n^2)}$$

$$x(t) = Ae^{-t} + B\cos\omega_n t + \frac{C}{\omega_n}\sin\omega_n t$$

$$= \frac{1}{1+\omega_n^2}\left[e^{-t} - \cos\omega_n t + \frac{1}{\omega_n}\sin\omega_n t\right]$$

CONVOLUTION

(4) $\ddot{x} + \omega_n^2 x = e^{-t}$ $\qquad x(0) = 0 \quad \dot{x}(0) = 0$

$$x(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\tau} \frac{1}{\omega_n} \sin \omega_n (t-\tau)\, dt$$

$$\begin{array}{ll} \frac{1}{\omega_n} \sin \omega_n (t-\tau) & e^{-\tau} \\ -1 \cos \omega_n (t-\tau) & -e^{-\tau} \\ -\omega_n \sin \omega_n (t-\tau) & e^{-\tau} \end{array}$$

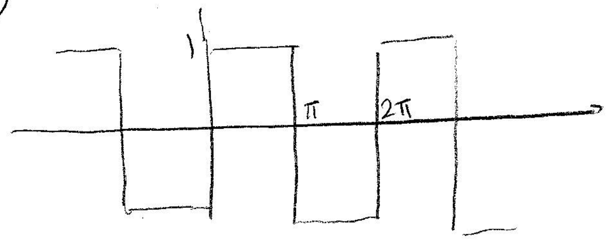
$$= -\frac{e^{-\tau}}{\omega_n} \sin \omega_n (t-\tau) + e^{-\tau} \cos \omega_n (t-\tau) \Big|_{\tau=0}^{\tau=t} - \omega_n \int_0^t e^{-\tau} \sin \omega_n (t-\tau)\, d\tau$$

$$\Rightarrow (1+\omega_n^2) \int_0^t e^{-\tau} \cdot \frac{1}{\omega_n} \sin \omega_n (t-\tau)\, d\tau = -\frac{e^{-\tau}}{\omega_n} \sin \omega_n (t-\tau) + e^{-\tau} \cos \omega_n (t-\tau) \Big|_0^t$$

$$x(t) = \frac{1}{(1+\omega_n^2)} \left[\cancel{-\frac{e^{-t}}{\omega_n} \sin(0)} + \frac{e^0}{\omega_n} \sin \omega_n t + e^{-t} \cos(0) - e^0 \cos \omega_n t \right]$$

$$= \frac{1}{(1+\omega_n^2)} \left[e^{-t} - \cos \omega_n t + \frac{1}{\omega_n} \sin \omega_n t \right]$$

⑤ Determine the Fourier Series for the rectangular wave



Period = $\tau = 2\pi$ Fundamental frequency: $\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2\pi} = 1$

a) Function is odd so a_n's $= 0$. Find b_n's only.

b) $$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\sin nt\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\sin nt\,dt$$

$$= \left.\frac{\cos nt}{-n\pi}\right|_0^{\pi} + \left.\frac{\cos nt}{+n\pi}\right|_{\pi}^{2\pi}$$

$$= \frac{1}{n\pi}\left[-\cos n\pi + \cos 0\right] + \frac{1}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$$

$$= \frac{2}{n\pi}\left[1 - \cos n\pi\right] = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

$$x(t) = \sum_{n=1,3,5,\ldots} \frac{4}{n\pi}\sin(nt)$$