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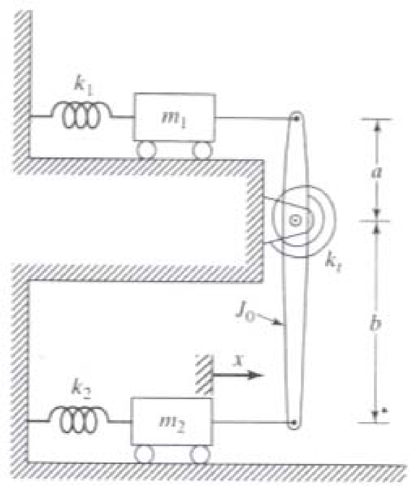
MCG 4303 Mechanical Vibration Analysis
Midterm Examination February 9 2009

Question 1

Consider the system shown in the figure below. For the given system,

(a) How many degrees of freedom does this system have?
(b) Using *x* as your variable, find the equivalent mass and stiffness of the system.
(c) Write the equation of motion of this system in terms of *x*.
(d) Find the natural frequency.
(e) If you were to add a damper *c* in parallel to the spring k_2, what value would *c* need to have to make the system critically damped?



Solution

Kinematics

$$\theta(t) = \frac{x(t)}{b}$$

$$x_1(t) = x(t) \cdot \frac{a}{b}$$

Writing the kinetic energy of the system

$$KE := \frac{1}{2}\cdot m_1\cdot\left(\frac{d}{dt}\,x_1(t)\right)^2 + \frac{1}{2}\cdot m_2\cdot\left(\dot{x}\right)^2 + \frac{1}{2}\cdot J_o\cdot\left(\dot{\theta}\right)^2$$

$$subs((4.1), (4.2), KE)$$

$$\frac{1}{2}\,\frac{\left(\frac{d}{dt}\,x(t)\right)^2\left(m_1\,a^2 + m_2\,b^2 + J_o\right)}{b^2}$$

$$KE2 := \frac{1}{2}\cdot m_{eq}\cdot\left(\frac{d}{dt}\,x(t)\right)^2$$

$$\frac{1}{2}\,m_{eq}\left(\frac{d}{dt}\,x(t)\right)^2$$

$$m_{eq} = solve((4.5) = (4.6), m_{eq})$$

$$m_{eq} = \frac{m_1\,a^2 + m_2\,b^2 + J_o}{b^2}$$

Write the elastic potential energy of the system

$$PE := \frac{1}{2}\cdot k_2\cdot x^2 + \frac{1}{2}\cdot k_t\cdot\theta^2 + \frac{1}{2}\cdot k_1\cdot x_1^2$$

$$subs\left(\theta = \frac{x}{b}, x_1 = x\cdot\frac{a}{b}, PE\right)$$

$$\frac{1}{2}\,\frac{x^2\left(k_2\,b^2 + k_t + k_1\,a^2\right)}{b^2}$$

$$PE2 := \frac{1}{2}\cdot k_{eq}\cdot x^2$$

$$k_{eq} = solve((4.10) = (4.11), k_{eq})$$

$$k_{eq} = \frac{k_2\,b^2 + k_t + k_1\,a^2}{b^2}$$

Equation of motion is given by

$$m_{eq} \cdot \ddot{x} + k_{eq} \cdot x = 0$$

Natural frequency is given by

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

$subs((4.7), (4.12), (4.14))$

$$\omega_n = \sqrt{\frac{k_2 b^2 + k_t + k_1 a^2}{m_1 a^2 + m_2 b^2 + J_o}}$$

To make this system critically damped, need to add a damper so that an additional term $-c \cdot \dot{x}$ is added to the equation of motion. This can be done by adding a damper in parallel with k_2 which has value $c_{crit} = 2 \cdot \sqrt{m_{eq} \cdot k_{eq}}$

$$c_{crit} = 2 \sqrt{m_{eq} k_{eq}}$$

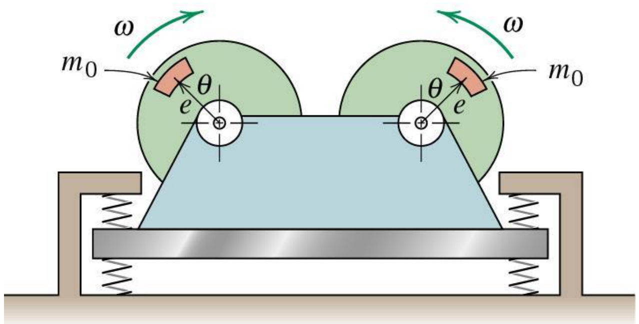
$subs((4.7), (4.12), (4.16))$

$$c_{crit} = 2 \sqrt{\frac{\left(m_1 a^2 + m_2 b^2 + J_o\right) \left(k_2 b^2 + k_t + k_1 a^2\right)}{b^4}}$$

Question 2

A device to produce vibrations consists of the two counter-rotating wheels, each carrying an eccentric mass $m_0 = 1$ kg with centre of mass at a distance $e = 12$ mm from its axis of rotation. The wheels are synchronized so that the vertical positions of the unbalanced masses are always identical. The total mass of the device is 10 kg. Neglect damping. The equivalent spring constant for all the springs combined together as shown is *k*.

(a) Write the equation of motion that describes the vibrations of this system.
(b) What is the response of the system to the rotation of the unbalanced masses?
(c) What force will be transmitted to the fixed mounting?
(d) Determine the two possible values of the equivalent spring constant *k* which will permit the *magnitude* of the periodic force transmitted to the fixed mounting to be 1500 N due to the imbalance of the rotors at a speed of 1800 rev/min.



Equation of motion is

$$m \cdot \ddot{x} + k \cdot x = 2 \cdot m_0 \cdot R \cdot \omega^2 \cdot \sin(\omega \cdot t)$$

Response will be of the form $x(t) = X \cdot \sin(\omega \cdot t)$.

$simplify(subs(x(t) = X \cdot \sin(\omega \cdot t), \mathbf{(47)}))$

$$m \left(\frac{\partial^2}{\partial t^2} (X \sin(\omega t)) \right) + k X \sin(\omega t) = 2 m_0 R \omega^2 \sin(\omega t)$$

$simplify\left(\frac{\mathbf{(48)}}{\sin(\omega \cdot t)} \right)$

$$X (k - m \omega^2) = 2 m_0 R \omega^2$$

$X = solve(\mathbf{(49)}, X)$

$$X = \frac{2 m_0 R \omega^2}{k - m \omega^2}$$

Force will be given by $F = k \cdot x(t) = k \cdot X \cdot \sin(\omega \cdot t)$

Force magnitude is $F = k \cdot X$

$$F = X k$$

$subs(\mathbf{(50)}, \mathbf{(51)})$

$$F = \frac{2 m_0 R \omega^2 k}{k - m \omega^2}$$

Substitute for $F = 1500$ **AND** then $F = -1500$ and solve for the corresponding value of k

$subs\left(m_0 = 1, R = 0.012, m = 10, F = 1500, \omega = evalf\left(\frac{1800 \cdot 2 \cdot \pi}{60} \right), \mathbf{(52)} \right)$

$$1500 = \frac{852.7338202 k}{k - 3.55305758410^5}$$

$solve(\mathbf{(53)}, k)$

$$8.23399482710^5$$

$$subs\left(m_0 = 1, R = 0.012, m = 10, F = -1500, \omega = evalf\left(\frac{1800 \cdot 2 \cdot \pi}{60}\right), \mathbf{(52)}\right)$$

$$-1500 = \frac{852.7338202 k}{k - 3.553057584 10^5}$$

$solve(\mathbf{(55)}, k)$

$2.265273840 10^5$

Question 3

A one degree-of-freedom mass-spring-damper system has effective properties of m = 16 kg, c = 8 Ns/m and k = 17 N/m. Initial conditions are given by $x_0 = 0, v_0 = 16$ m/s .

(a) Write the equation of motion and find the free vibration response of this system.
(b) Is this system underdamped, overdamped or critically damped?
(c) What are the natural frequency and damped frequency of this system?
(d) Suppose that the system is forced with a forcing function given by $F(t) = a_1 \cos(\omega t) + a_2 \cos(2\omega t)$. If you neglect damping, for what value(s) of the input frequency ω do you expect to observe excessively large vibrations?
(e) Do you expect your answer to part (d) to change if you include the given system damping? Why or why not?

A mass-spring-damper system has properties of $m = 16, c = 8, k = 17$. Initial conditions are $x_0 = 0, v_0 = 16$.

Equation of motion is

$$m\left(\frac{d^2}{dt^2} x(t)\right) + c\left(\frac{d}{dt} x(t)\right) + k\, x(t) = 0$$

Characteristic polynomial

$16 s^2 + 8 s + 17 = 0$

$solve(\mathbf{(27)})$

$-\frac{1}{4} + I, -\frac{1}{4} - I$

$y(t) := a_1 \cdot \exp\left(-\frac{1}{4} \cdot t\right) \cdot \sin(t) + a_2 \cdot \exp\left(-\frac{1}{4} \cdot t\right) \cdot \cos(t)$

$t \to a_1\, e^{-\frac{1}{4} t} \sin(t) + a_2\, e^{-\frac{1}{4} t} \cos(t)$

$y(0) = 0$

$a_2 = 0$

$v(t) := \frac{d}{dt} y(t)$

$$-\frac{1}{4} a_1 e^{-\frac{1}{4}t} \sin(t) + a_1 e^{-\frac{1}{4}t} \cos(t) - \frac{1}{4} a_2 e^{-\frac{1}{4}t} \cos(t) - a_2 e^{-\frac{1}{4}t} \sin(t)$$

$v(t)\Big|_{t=0} = 16$

$a_1 - \frac{1}{4} a_2 = 16$

$dsolve(\{16\cdot \ddot{x} + 8\cdot \dot{x} + 17\cdot x = 0, x(0) = 0, \dot{x}(0) = 16\})$

$x(t) = 16 e^{-\frac{1}{4}t} \sin(t)$

From the solution, can see that $\omega_d = 1$

→ system is clearly underdamped (can see this from solution – no need to calculate ς.

$\omega_n = \sqrt{\frac{k}{m}}$

$\zeta = \frac{c}{2\cdot\sqrt{m\cdot k}}$

$\omega_d = \sqrt{\frac{k}{m}} \cdot \sqrt{1 - \frac{c^2}{4\cdot m\cdot k}}$

$simplify(subs(m = 16, c = 8, k = 17, \mathbf{(36)}))$

$\omega_n = \frac{1}{4}\sqrt{17} \xrightarrow{\text{at 5 digits}} \omega_n = 1.0308$

$evalf(simplify(subs(m = 16, c = 8, k = 17, \mathbf{(37)})))$

$\zeta = 0.242535625$

$simplify(subs(m = 16, c = 8, k = 17, \mathbf{(38)}))$

$\omega_d = 1$

Natural frequency and damped frequency are pretty close together.

If input is given by $F = a_1\cdot\cos(\omega\cdot t) + a_2\cdot\cos(2\cdot\omega\cdot t)$ then potential trouble spots are at $\omega \approx \omega_n$ and $2\cdot\omega = \omega_n$ if we ignore the damping. Since the damping is pretty small and the natural and damped frequencies are pretty similar, we don't expect this answer to be that much different if damping is included.

$\omega_{criti} = evalf(simplify(subs(m = 16, c = 8, k = 17, \mathbf{(36)})))$

$\omega_{criti} = (\omega_n = 1.030776406)$

$\omega_{criti} = \frac{1}{2}\cdot evalf(simplify(subs(m = 16, c = 8, k = 17, \mathbf{(36)})))$

$\omega_{criti} = \left(\frac{1}{2}\omega_n = 0.5153882030\right)$

Because the damping is small, expect that actual maximum of the frequency response is not exactly the same as the natural frequency but probably pretty close.

You weren't expected to calculate this for the exam, but FYI we can see the effect of including damping:

Frequency Response

$$M = \frac{1}{k} \cdot \frac{1}{\sqrt{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2}}$$

$Mag := rhs(subs(\mathbf{(40)}, k = 17, \mathbf{(44)}))$

$$\frac{1}{17\sqrt{1 - 1.764705882 r^2 + r^4}}$$

$with(Optimization):$

$Maximize(\mathbf{(45)})$

$[0.124999999912109388, [r = 0.939336436534130458]]$

So the actual values of frequency that cause the magnitude to be largest are

$$0.939336436462338330 \frac{1}{4} \sqrt{17} = 0.2348341091 \sqrt{17} \xrightarrow{\text{at 5 digits}} 0.9682$$

$$\frac{1}{2} \cdot 0.939336436462338330 \frac{1}{4} \sqrt{17} = 0.1174170546 \sqrt{17} \xrightarrow{\text{at 5 digits}} 0.4841$$

These numbers are slightly different from what we got by using the natural frequency itself as the "worst case" but not that far off. I.e. for low damping, can use the natural frequency as a rough guideline for where the maximum amplitude is. It's NOT exact, but it is quick to compute.