Université d'Ottawa
Faculté de genie

Départment de
Génie Méchanique

uOttawa
L'Université canadienne
Canada's university

University of Ottawa
Faculty of Engineering

Department of
Mechanical Engineering

# MCG 4308
# Mechanical Vibration Analysis

**MIDTERM EXAMINATION**
**February 18, 2011**

**Comment [d1]:** Total on 30 points

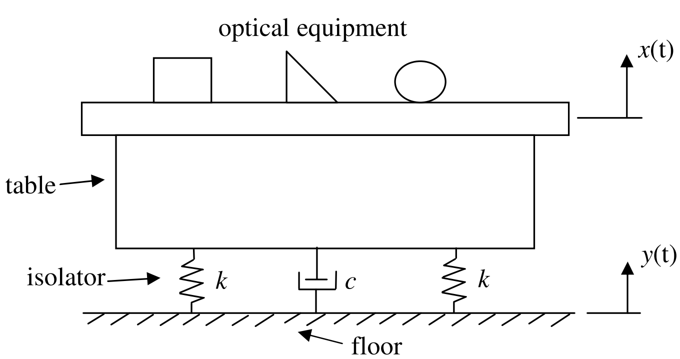
Comments and general instructions:

1. PUT YOUR NAME AND STUDENT NUMBER ON YOUR EXAM BOOKLET!
2. There are ***three problems*** of ***equal weight***.
3. Potentially useful mathematical formulae are on the last two pages.
4. Be neat and clear in your presentation, and show all work for full credit.
5. This exam is closed book.
6. Time: **75 minutes**.

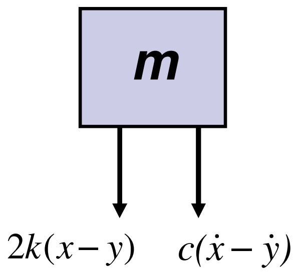
**Question 1**

A table on which is mounted extremely sensitive optical equipment is supported on an isolator consisting of two springs and a damper. The floor, on which the machine is mounted, is subjected to a harmonic disturbance due to the operation of an unbalanced pump that is located nearby.

1. Write the equation of motion for this system. **Comment [d2]:** 3 points
2. Calculate the displacement transmissibility of the base motion. (Hint: Assume $x = Xe^{i(\omega t+\phi)}$ and $y = Ye^{i\omega t}$) **Comment [d3]:** 3 points
3. If $k = 625$kN/m, $c = 50$kg/s and $m = 500$kg, what is the table's natural frequency? **Comment [d4]:** 3 points
4. The optical equipment has the ability to resist an acceleration of up to 3.5g without being affected. Calculate the maximum acceleration acting on the equipment due to the unbalanced pump if the pump is causing the floor to shake with a displacement of $y(t) = 0.002\cos(53t)$. **Comment [d5]:** 3 points
5. It is discovered that the operating frequency of the unbalanced pump is close to the natural frequency of the table so that the displacement of the optical equipment is too large. Which of the following techniques can be used, with confidence, to reduce the motion of the table? Choose ALL that apply and explain briefly. **Comment [d6]:** 3 points
   a. Reduce the mass of the optical table
   b. Increase the mass of the optical table
   c. Reduce the stiffness of the isolator (springs)
   d. Increase the stiffness of the isolator (springs)
   e. Reduce the damping of the isolator
   f. Increase the damping of the isolator



**Solution**




1. Equation: $m\ddot{x}+c\dot{x}+2kx=c\dot{y}+2ky$

**+3pt**

or $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=2\zeta\omega_n\dot{y}+\omega_n^2y \qquad \omega_n^2=\frac{2k}{m}$

2. Sub into equation: $y(t)=Ye^{i\omega t} \qquad x(t)=Xe^{i(\omega t+\phi)}$ **+1pt**

or $y(t)=Y\sin(\omega t) \quad x(t)=X\sin(\omega t+\phi)$ etc....

$$\left[-\omega^2+2\zeta\omega_n\omega i+\omega_n^2\right]Xe^{i(\omega t+\phi)}=\left[2\zeta\omega_n\omega i+\omega_n^2\right]Ye^{i\omega t}$$

or $\left[-m\omega^2+c\omega i+k\right]Xe^{i(\omega t+\phi)}=\left[c\omega i+k\right]Ye^{i\omega t}$

Divide both sides by $\omega_n^2$ and remember $r=\omega/\omega_n$

$$\left[-r^2+2\zeta ri+1\right]Xe^{i(\omega t+\phi)}=\left[2\zeta ri+1\right]Ye^{i\omega t}$$

$$\frac{Xe^{i(\omega t+\phi)}}{Ye^{i\omega t}}=\frac{X}{Y}e^{i\phi}=\frac{2\zeta ri+1}{1-r^2+2\zeta ri}$$ **+1pt**

So match up magnitudes:

$$\frac{X}{Y}=\text{magnitude of}\left[\frac{2\zeta ri+1}{1-r^2+2\zeta ri}\right]=\sqrt{\frac{(2\zeta r)^2+1}{(1-r^2)^2+(2\zeta r)^2}}$$

$$\boxed{\frac{X}{Y}=\sqrt{\frac{(2\zeta r)^2+1}{(1-r^2)^2+(2\zeta r)^2}}} \quad \text{or} \quad \frac{X}{Y}=\sqrt{\frac{(2k)^2+(c\omega)^2}{(2k-m\omega^2)^2+(c\omega)^2}}$$

Displacement Transmissibility Ratio **+1pt**

3.

$$\omega_n^2=\frac{2k}{m}$$ **+2pt**

$$\omega_n=\sqrt{\frac{2\times 625000}{500}}=50\,\text{rad/s}$$ **+1pt**

4.

$$\ddot{x}_{max}=\left|-\omega_n^2X\right| \quad \text{where} \quad X=Y\sqrt{\frac{(2\zeta r)^2+1}{(1-r^2)^2+(2\zeta r)^2}} \quad \text{and} \quad \zeta=\frac{c}{2\sqrt{(2k)m}}$$ **+1pt**

$$\zeta=\frac{c}{2\sqrt{(2k)m}}=\frac{50}{2\sqrt{2\times 625000\times 500}}=0.001$$

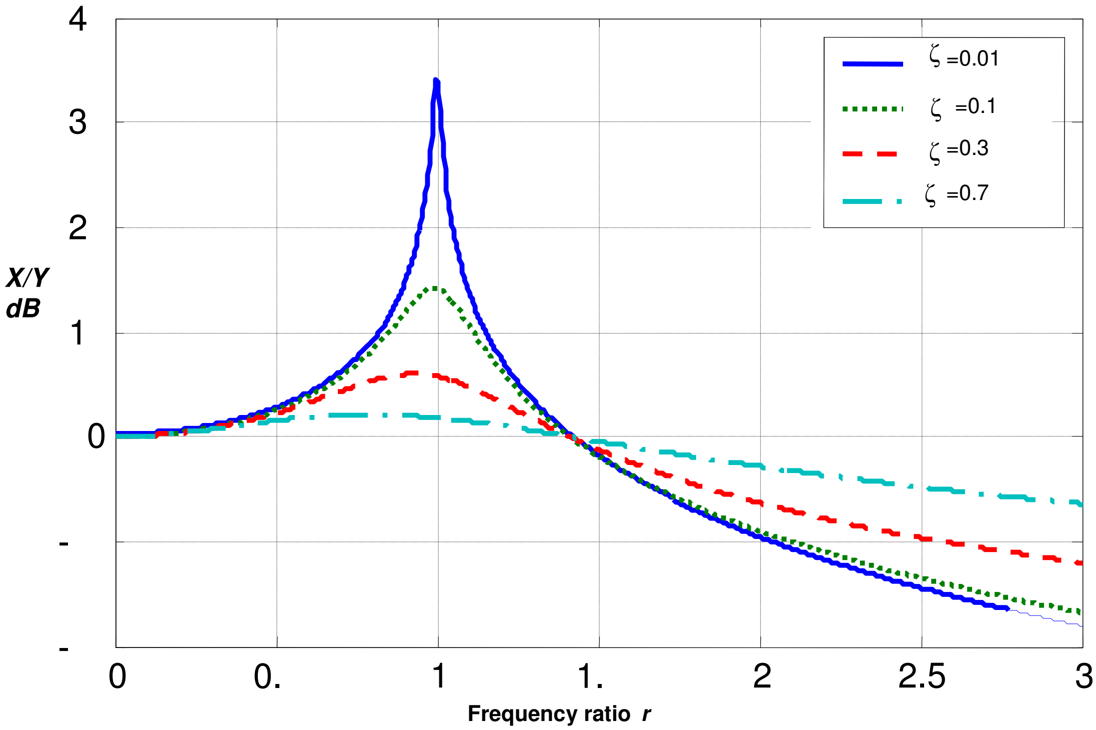
**+1pt**

$$r=\frac{\omega}{\omega_n}=\frac{53}{50}=1.06$$

$$X=0.002\sqrt{\frac{(2\times 0.001\times 1.06)^2+1}{(1-1.06^2)^2+(2\times 0.001\times 1.06)^2}}=0.016$$ **+1pt**

$$\ddot{x}_{max}=\left|-50^2\times 0.016\right|=40\,\text{m/s}^2 \quad \text{or} \quad 4.1\,\text{g}$$

5.




To decrease amplitude of response, need to increase damping or change the natural frequency of the system. Changing either the mass or stiffness will change the natural frequency of the system. So all situations except for (e) apply. We can change the natural frequency (change m or k) or increase the damping.

**+3pt**

**Comment [d7]:** If they get portions right, give them partial marks.

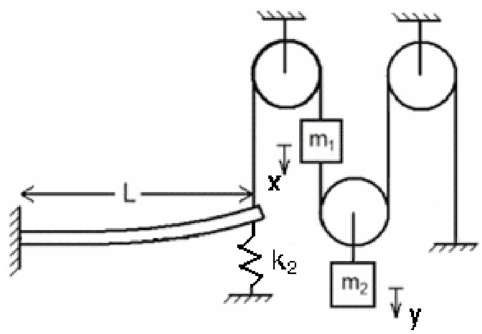
**Question 2**

Consider the system shown in the figure. Use $y$(t), the vertical displacement of $m_2$ from the system's equilibrium position, as the generalized coordinate. The stiffness of the cantilevered beam can be measured as $k_1 = \frac{Ewh^3}{4L^3}$ where $w$ is the width and $h$ the height of the cross section of the beam. For this system find:

1. The equivalent mass
2. The equivalent stiffness (hint: use springs in parallel)

Comment [d8]: 3 points

Comment [d9]: 3 points




**Solution**

Relationship between $x$ and $y$ $\Rightarrow$ $x = 2y$ **+1pt**

1.

Kinetic energy

$$T = \frac{1}{2}m_1 x^2 + \frac{1}{2}m_2 y^2 = \frac{1}{2}m_1(2y)^2 + \frac{1}{2}m_2 y^2 = \frac{1}{2}4m_1 y^2 + \frac{1}{2}m_2 y^2 = \frac{1}{2}(4m_1 + m_2)y^2$$ **+1pt**

$\therefore \; m_{eq} = 4m_1 + m_2$ **+1pt**

2.

Springs in parallel

$k_{tot} = k_1 + k_2$ where $k_1 = \frac{Ewh^3}{4L^3}$ **+1pt**

If they used in series because they didn't listen to correction that's ok but answer is this:

$$\frac{1}{k_{tot}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_2 + k_1}{k_1 k_2}$$

$$k_{tot} = \frac{k_1 k_2}{k_1 + k_2} \quad \text{where} \quad k_1 = \frac{Ewh^3}{4L^3}$$

Potential energy

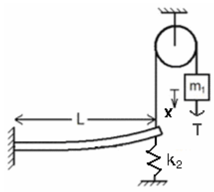
$$V = \frac{1}{2}k_{tot}x^2 = \frac{1}{2}k_{tot}(2y)^2 = \frac{1}{2}4k_{tot}y^2$$ **+1pt**

$k_{eq} = 4k_{tot}$ **+1pt**

Or using Newton's method

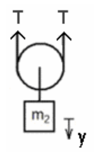
From diagram A

Comment [d10]: If they use Newton's method instead, allocate points similarly to the energy method above.




$$\sum F = m_1\ddot{x} = -k_{tot}x + T$$
$$T = m_1\ddot{x} + k_{tot}x$$
$$T = 2m_1\ddot{y} + 2k_{tot}y$$

From diagram B




$$\sum F = m_2\ddot{y} = -2T$$
$$-2T = m_2\ddot{y}$$

By making the tension equal we get

$$-2\left(2m_1\ddot{y} + 2k_{tot}y\right) = m_2\ddot{y}$$
$$\underbrace{\left(4m_1 + m_2\right)}_{m_{eq}}\ddot{y} + \underbrace{4k_{tot}}_{k_{eq}}y = 4$$

**<u>Question 3</u>**

The motion of a spring-mass-damper system is described by the following equation:

$$\ddot{x} + 16x = 48 + 20e^{2t}$$

If this system is initially located at a displacement of 4 with zero initial velocity, find:

1. The total (homogeneous and particular) solution to the problem
2. The steady-state solution to the problem

**Comment [d11]:** 3 points

**Comment [d12]:** 3 points

**Comment [d13]:** 3 points

**<u>Solution</u>**

1.

General form for *x*:

$$x(t) = x_h(t) + x_{p1}(t)\Big|_{f_1=48} + x_{p2}(t)\Big|_{f_2=20e^{2t}}$$

$\ddot{x}+16x=48+20e^{2t} \qquad x(0)=4, \quad \dot{x}(0)=0$

Guess and Check:

$x(t)=x_h(t)+x_{p1}(t)\big|_{f_1=48}+x_{p2}(t)\big|_{f_2=20e^{2t}}$

Homogeneous Solution: $\ddot{x}_h+16x_h=0$ **+1pt**

Guess $x_h=e^{st}$ and sub into equation : $\left[s^2+16\right]e^{st}=0$

characteristic equation: $s^2+16$, with roots: $s=\pm 4i$ **+1pt**

$\Rightarrow x_h(t)=A_1\cos(4t)+A_2\sin(4t)$

First particular solution: $\ddot{x}_{p_1}+16x_{p_1}=48$

RHS is a constant so guess $x_{p_1}=C$ (constant). Sub into equation, clearly $x_{p_1}=3$.

Second particular solution: $\ddot{x}_{p_2}+16x_{p_2}=20e^{2t}$

Guess $x_{p_2}(t)=Be^{2t}$ where $\ddot{x}_{p_2}=4Be^{2t}$ and sub into equation: **+1pt**

$4Be^{2t}+16\left(Be^{2t}\right)=20e^{2t}$

$(4B+16B)\cancel{e^{2t}}=20\cancel{e^{2t}} \qquad \Rightarrow \qquad 20B=20 \qquad \Rightarrow \qquad B=1$ **+1pt**

Therefore, $x_{p_2}=e^{2t}$

so: $x(t)=x_h(t)+x_{p1}(t)\big|_{f_1=48}+x_{p2}(t)\big|_{f_2=20e^{2t}}$

$=A_1\cos(4t)+A_2\sin(4t)+3+e^{2t}$ **+1pt**

$\dot{x}(t)=-4A_1\sin(4t)+4A_2\cos(4t)+2e^{2t}$

Now substitute initial conditions to find $A_1$ and $A_2$

$x(0)=A_1+3+1=4 \quad \Rightarrow \quad A_1=0$

$\dot{x}(0)=4A_2+2=0 \quad \Rightarrow \quad A_2=-0.5$

Finally, total solution is:

$x(t)=A_1\cos(4t)+A_2\sin(4t)+3+e^{2t}$ **+1pt**

$=3-0.5\sin(4t)+e^{2t}$

2.
The steady state solution is the part of the solution that doesn't die or explode. If part of the solution dies, then what is left is the steady state solution. If part of the solution explodes then we are left with no solution, therefore the steady state solution is what is left if we get rid of the exploding part. The exponential term is the only term that explodes, so that the steady state solution is given by **+2pt**

**Comment [d14]:** If they explain something that make sense don't be afraid to give them points or partial points on this.

$$x_{ss}=3-0.5\sin(4t)$$ **+1pt**

## Useful formulas

Quadratic formula: roots of $ax^2+bx+c$ are given by $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Integration by parts: $\int udv = uv - \int vdu$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\sin(u) = \dfrac{e^{iu}-e^{-iu}}{2i} = \mathrm{Im}\{e^{iu}\}$ $\qquad \cos(u) = \dfrac{e^{iu}+e^{-iu}}{2} = \mathrm{Re}\{e^{iu}\}$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$$a_0 = \frac{2}{T}\int_0^T x(t)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2n\pi t}{T}\right)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2n\pi t}{T}\right)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t)\sin(n\omega t)dt$$

**Small angle approximations:** $\begin{cases} \sin\theta \approx \theta \\ \cos\theta \approx 1-\dfrac{\theta^2}{2} \end{cases}$

| **2nd Order Linear Homogeneous ODE with Constant Coefficients**: $ay'' + by' + cy = 0$ | | |
|---|---|---|
| **Characteristic Equation**: $a\lambda^2 + b\lambda + c = 0$ | | |
| **Roots of Characteristic Equation** $\lambda_1$ and $\lambda_2$ | | **General Solution** |
| 1 | $\lambda_1\ \lambda_2 \in \Re,\ \lambda_1 \neq \lambda_2$ | $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ |
| 2 | $\lambda_1\ \lambda_2 \in \Re,\ \lambda_1 = \lambda_2 = \lambda$ | $y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$ |
| 3 | $\lambda_1 = \sigma + \omega i,\ \lambda_2 = \sigma - \omega i$<br>where $\sigma, \omega \in \Re$ | $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$<br>$= d_1 e^{\sigma t}\cos(\omega t) + d_2 e^{\sigma t}\sin(\omega t)$ |