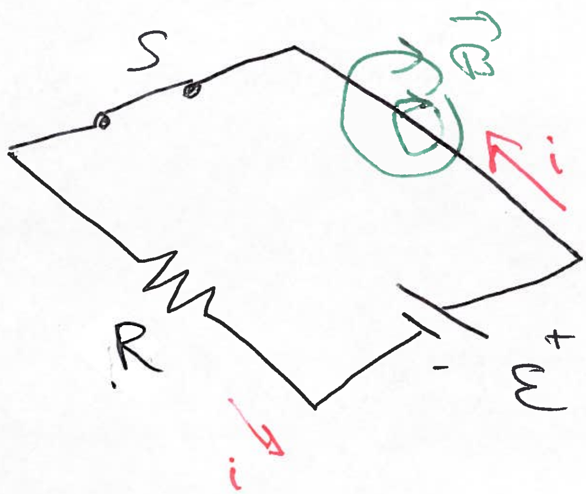
Inductance Chapter 32

Self induction and inductance



When the switch is closed $\Rightarrow i = \mathcal{E}/R$

$\Rightarrow$ not immediately

$i \nearrow \Rightarrow B \nearrow \Rightarrow$ change in Φ_B over the loop of the circuits

$\Rightarrow$ induced emf in the circuit

$\Rightarrow$ induced current would create $\vec{B}_{ind}$ opposing to the change in $\vec{B}$

induced emf is opposed to the original emf of the battery

$\Rightarrow$ consequence : gradual increase of i

This is the same phenomenon as the back emf in a motor.

$\Rightarrow$ $\mathcal{E}_L$: self. induced emf

$$\mathcal{E}_L = -L \frac{di}{dt}$$

proportional to the time rate of change of i

L : proportionality constant = $\boxed{\text{inductance}}$

$\hookrightarrow$ geometry of the loop + physical characteristics

from Faraday's law $\mathcal{E}_L = -N \frac{d\Phi_B}{dt}$

N loops

$\Rightarrow$ $L\, di = N\, d\Phi_B$

$$\Rightarrow \left\| L = \frac{N\,\Phi_B}{i} \right. \quad , \quad L = -\frac{\mathcal{E}_L}{di/dt}$$

cf: $R = \Delta V / I$

R: measure of opposition to current

L: measure of opposition to a change in current

SI = 1 henry = 1 H = 1 V.s/A

Inductance of a solenoid

Uniform solenoid of N turns and length ℓ

$\ell \gg$ radius of the cylinder

$\Rightarrow \vec{B}$ uniform inside the solenoid

$$\Rightarrow B = \mu_0 n i = \mu_0 \frac{N}{\ell} i$$

$$\Phi_B = \oint_S \underbrace{\vec{B}}_{\text{uniform}} \cdot d\vec{S} = B \underbrace{\oint dS}_{S} = \mu_0 \frac{N}{\ell} i S$$

$$\Rightarrow \boxed{L = \frac{N\Phi_B}{i} = \mu_0 \frac{N^2}{\ell} S}$$

Note that $N = n\ell$

$$\Rightarrow L = \mu_0 \frac{(n\ell)^2}{\ell} S = \mu_0 n^2 \underbrace{S\ell}_{\text{Volume of the solenoid } V}$$

$$\Rightarrow \boxed{L = \mu_0 n^2 V}$$

RL circuits

Inductance of the wire $\Rightarrow$ <u>neglect</u>

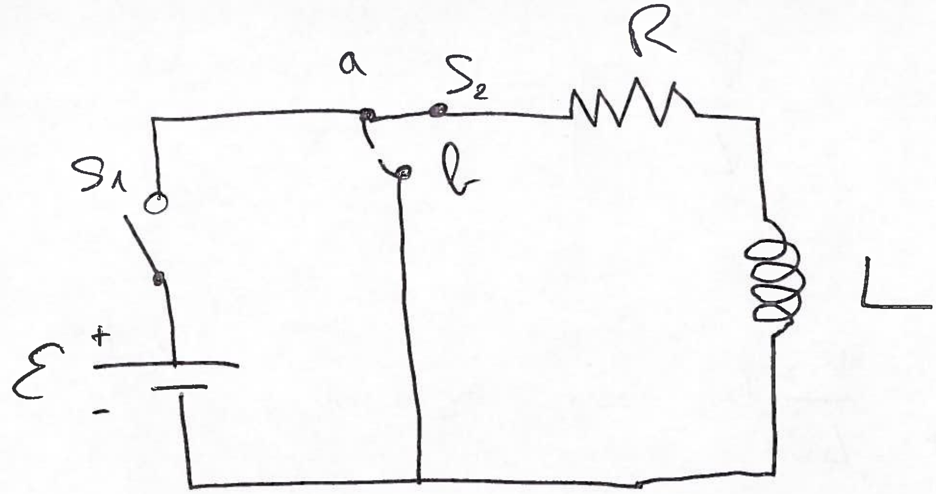
Element of circuit with large inductance

$\Rightarrow$ <u>Inductor</u> 

L

$\Rightarrow$ "sluggish" circuits.

L3



S_1 open for $t < 0$, closed at $\underline{\underline{t=0}}$

$i \nearrow$ and a back emf is created in the solenoid.

$\Rightarrow$ Kirchhoff's rule

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

To solve this $\Rightarrow$ change of variable

$$x = \mathcal{E}/R - i \quad \Rightarrow \quad dx = -di$$

$$\Rightarrow \quad x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\int_{x_0}^{x} \frac{dx}{x} = -\frac{R}{L}\int_0^t dt$$

$$\ln\frac{x}{x_0} = -\frac{R}{L}t \Rightarrow x = x_0 e^{-R/L\, t}$$

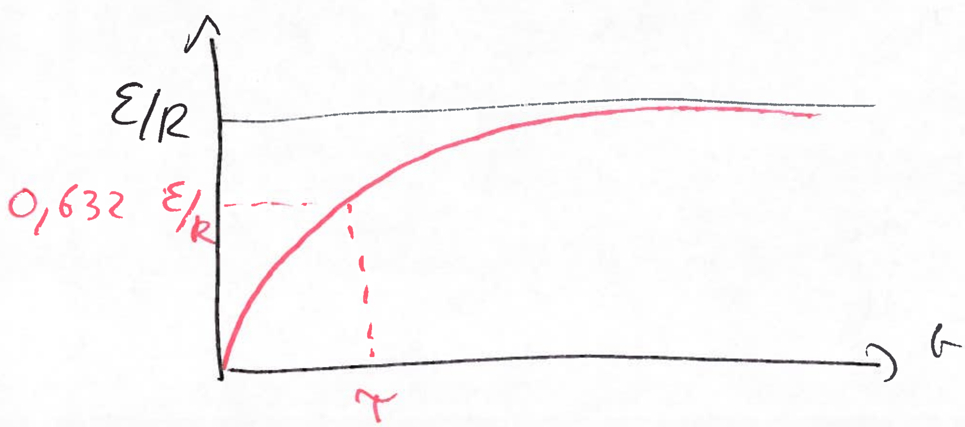
at $t=0$, $i=0 \Rightarrow x = \mathcal{E}/R$

$$\Rightarrow \frac{\mathcal{E}}{R} - i = \frac{\mathcal{E}}{R} e^{-R/L\, t}$$

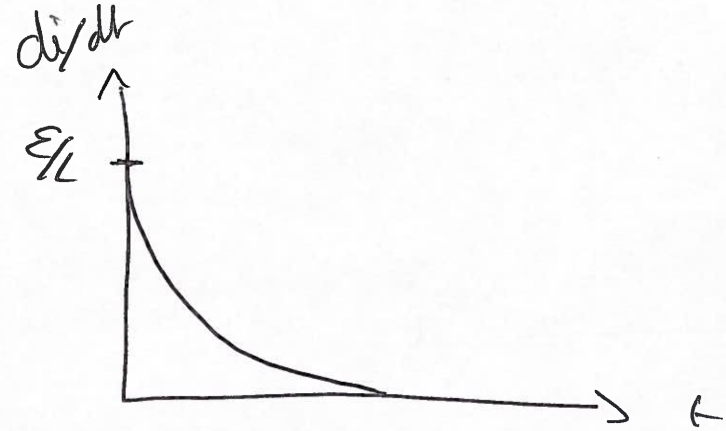
$$\Rightarrow \boxed{i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right)}$$

$$= \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

time constant $\tau = L/R$

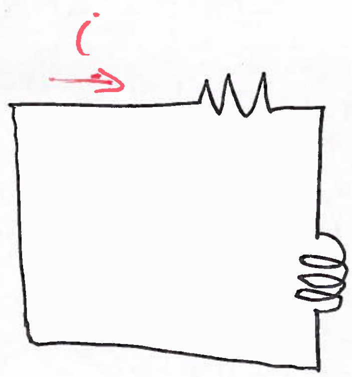


We have $\frac{di}{dt} = \frac{\mathcal{E}}{L} e^{-t/\tau}$



* $i = \mathcal{E}/R$ (equilibrium), S_2 switch from a to b

$\Rightarrow$ the battery is ell eliminate



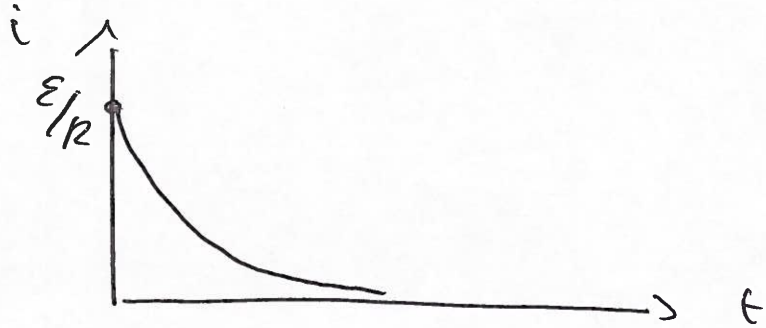
at $t = 0$, $i = \underbrace{\mathcal{E}/R}_{I_0}$

$$\Rightarrow \quad iR + L\frac{di}{dt} = 0$$

$$\Leftrightarrow \quad \frac{di}{i} = -\frac{R}{L}\,dt$$

$$i = I_0\, e^{-R/L\, t}$$

$$i(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}$$



Note if L really smal $\Rightarrow$ no inductance effect

$\Rightarrow \tau$ small

if no inductor $\Rightarrow$ i would increase ↑ to $\mathcal{E}/R$ or decrease ↓ to 0 immediately

$\Rightarrow$ the presence of L slows down the process

Energy in a magnetic field

$$\mathcal{E} = Ri + L\frac{di}{dt}$$ (S_2 on a, S_1 just closed)

$\times i$

$$\underbrace{i\mathcal{E}}_{\text{rate of energy supplied by the battery}} = \underbrace{Ri^2}_{\text{rate of energy (power) delivered to the resistor}} + Li\frac{di}{dt}$$

$U_B \Rightarrow$ energy stored in the inductor at any time

Rate of energy $\Rightarrow \dfrac{dU_B}{dt}$

$$\Rightarrow \frac{dU_B}{dt} = L\, i\, \frac{di}{dt}$$

$$\Rightarrow dU_B = L\, i\, di$$

$$\Rightarrow U_B = \int dU_B = \int_0^i L\, i\, di = L \underbrace{\int_0^i i\, di}_{1/2\, i^2}$$

$$\Rightarrow \boxed{U_B = \frac{1}{2} L\, i^2}$$

Energy stored in the magnetic field of the inductor

cf : energy stored in the electric field of a capacitor

$$U_E = \frac{1}{2} C (\Delta V)^2$$

For a solenoid : $L = \mu_0 n^2 \underbrace{V}_{\text{volume}}$

with $B = \mu_0 n i$

$$\Rightarrow U_B = \frac{1}{2} L i^2 = \frac{1}{2} \mu_0 \cancel{n^2} V \left(\frac{B}{\mu_0 \cancel{n}}\right)^2$$

$$= \frac{B^2}{2\mu_0} V$$

Magnetic energy density : $u_B = U_B / V$

$$\Rightarrow \boxed{u_B = \frac{B^2}{2\mu_0}}$$

cf: Energy per unit volume stored in an electric field

$$\Rightarrow u_E = \frac{1}{2} \varepsilon_0 E^2$$

What happens to the energy in the inductor?

$$i(0) = I_i = \mathcal{E}/R$$

at $t = 0$ S_2 switched to b

internal energy of the resistor $\frac{dE_{int}}{dt} = P = i^2 R$

with $i = \underbrace{I_i}_{\mathcal{E}/R} e^{-t/\tau} = I_i e^{-Rt/L}$

$$\Rightarrow \frac{dE_{int}}{dt} = \left(I_i e^{-Rt/L}\right)^2 R$$

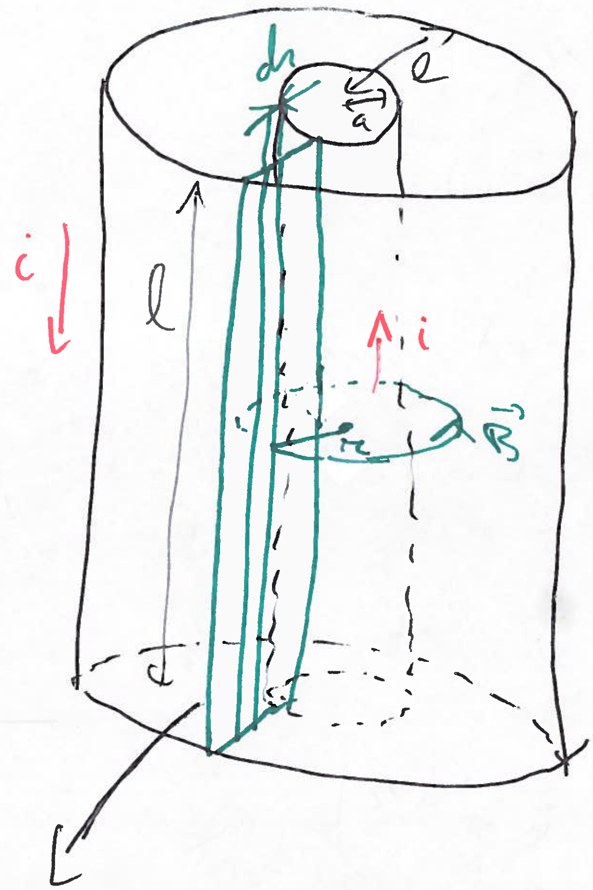
$$= I_i^2 R e^{-2Rt/L}$$

$$E_{int} = \int_0^{+\infty} I_i^2 R e^{-2Rt/L} dt$$

$$= R I_i^2 \int_0^{+\infty} e^{-2Rt/L} dt$$

$$= I^2 R \left(\frac{L}{2R}\right) = \frac{1}{2} L I_i^2 = \text{initial magnetic energy stored in the inductor}$$

Inductance Coaxial cable



Magnetic field between the two cable (Ampere law)

$\Downarrow$

depends on r only

$$\Rightarrow B = \frac{\mu_0 i}{2\pi r}$$

Magnetic field outside ($r > b$)

$B = 0$ (Ampere law)

Let's consider this loop "closed" at the extremity of the cable

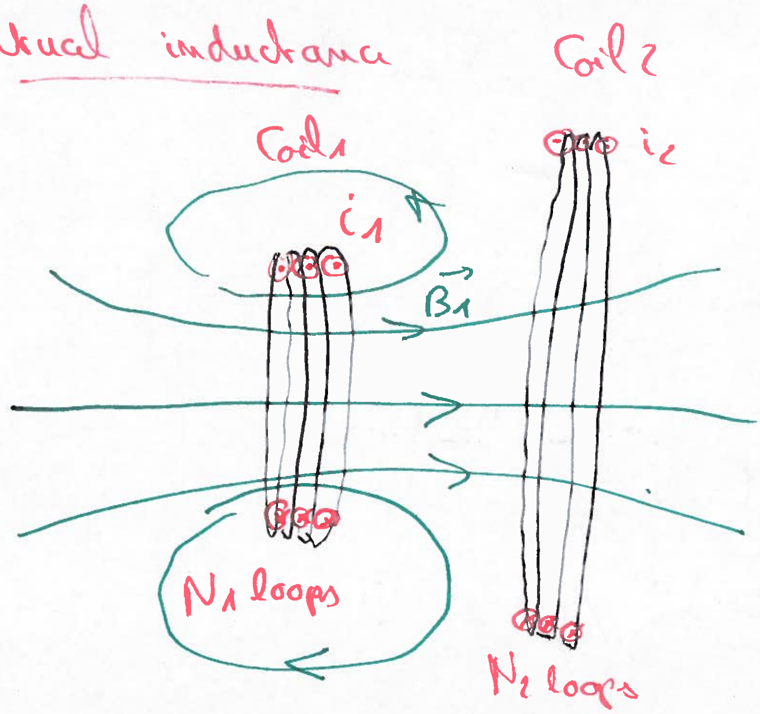
Flux through this loop

$$\phi_B = \int_S \vec{B} \cdot d\vec{S} = \int_a^b \frac{\mu_0 i}{2\pi r}\, l\, dr$$

$$= \frac{\mu_0 i l}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 i l}{2\pi} \ln(b/a)$$

By definition $L = \dfrac{\phi_B}{i} = \dfrac{\mu_0 l}{2\pi} \ln(b/a)$

Mutual inductance



Influence of the magnetic flux created by coil 1 on coil 2

Φ_{12} : flux of magnetic field $\vec{B}_1$ created by current i_1 in coil 1, in coil 2

Mutual inductance

$$M_{12} = \frac{N_2 \Phi_{12}}{i_1}$$

Faraday law on coil 2 :

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt}\left(\frac{M_{12}\, i_1}{N_2}\right)$$

$\Rightarrow$ $\| \mathcal{E}_2 = -M_{12}\, di_1/dt$ $\quad$ $\mathcal{E}_2$: emf induced on coil 2 by current in coil 1

Similarly : $\mathcal{E}_1 = -M_{21} \dfrac{di_2}{dt}$

↳ Mutual inductance : $M_{12} = M_{21} = M$

$\mathcal{E}_2 = -M\, di_1/dt$

$\mathcal{E}_1 = -M\, di_2/dt$

eg : "Wireless" battery charge

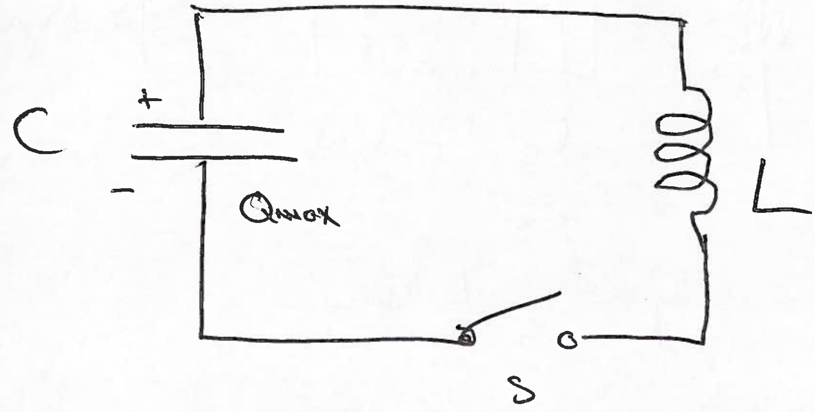
: $B = \dfrac{\mu_0 N_B}{\ell}\, i$

$$M = \frac{N_H\, \Phi_{BH}}{i} = \frac{N_H\, B\, S}{i}$$

$$M = \mu_0 \frac{N_H N_B}{\ell} S$$

⇒ used to charge wireless device

Oscillations in an LC circuits



C fully charged for $t < 0$

S open for $t < 0$ at $t = 0$ S = close

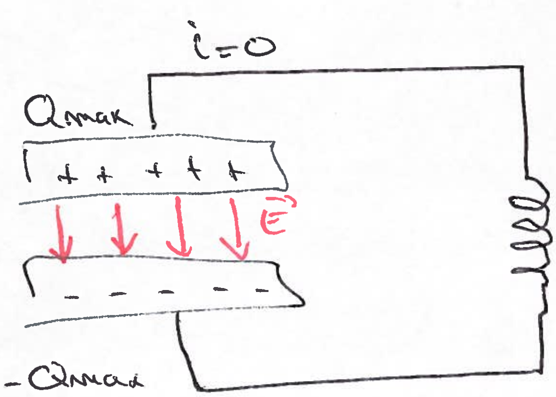
We assume : - no resistance, no energy transformed to internal energy

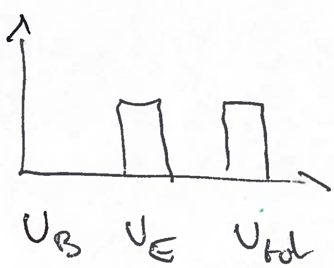
- no radiation

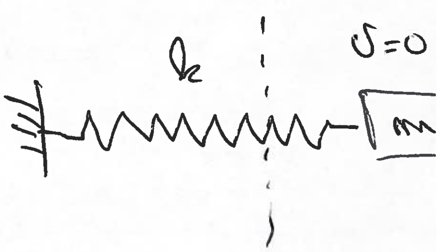
⟹ **Energy conservation**

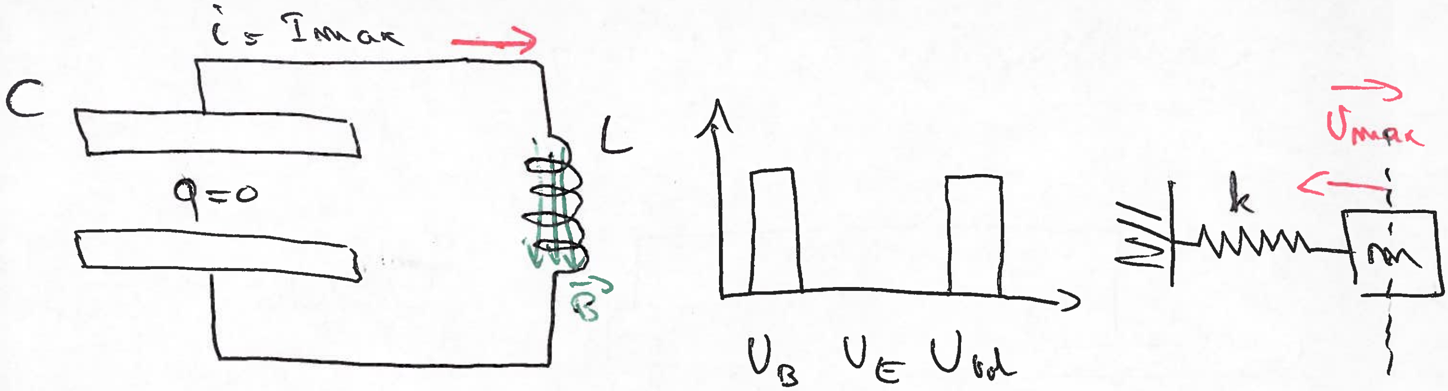
at $t = 0$

$$\left. \begin{array}{l} U_E = \dfrac{1}{2}\dfrac{Q_{max}^2}{C} \\ U_B = 0 \end{array} \right\} U_{tot} = U_E + U_B$$

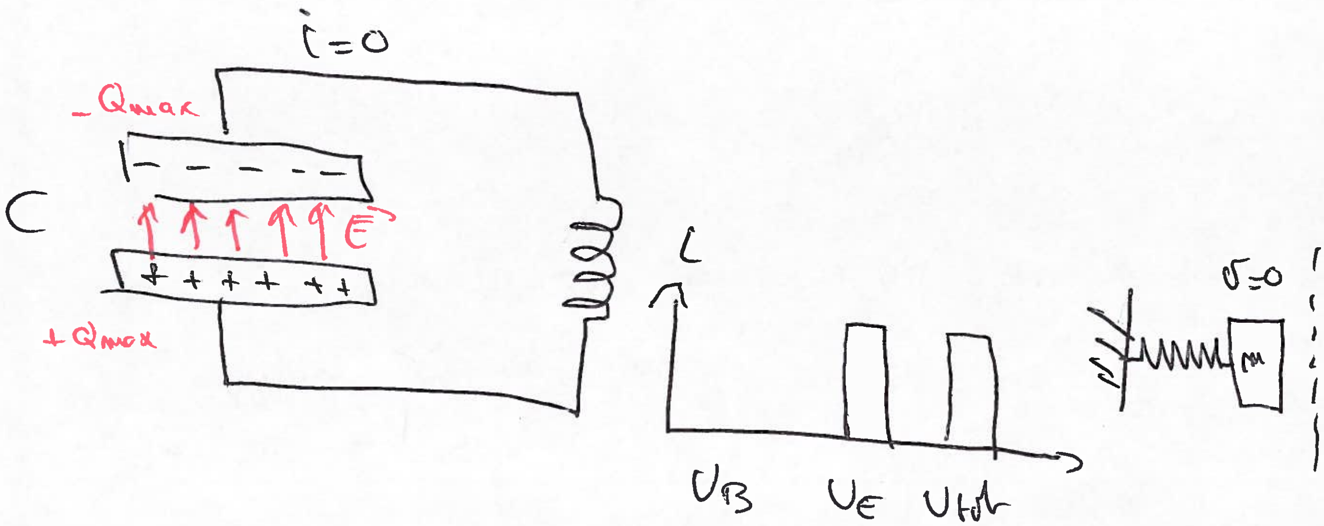


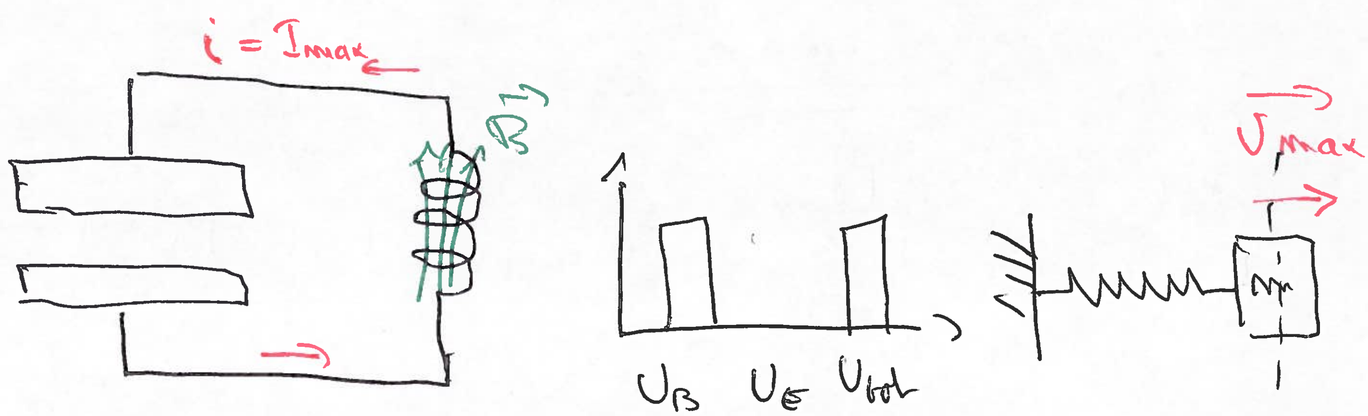






$$U_B = \frac{1}{2} L I_{max}^2$$





$\Longrightarrow$ <u>oscillation</u>

At any time

$$U_{tot} = U_E + U_B = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \underbrace{\frac{Q_{max}^2}{2C}}_{\text{initial condition}}$$

U_{tot} = constant $\Rightarrow \dfrac{dU_{tot}}{dt} = 0$

$$\frac{d}{dt}\left(\frac{q^2}{2C} + \frac{1}{2}Li^2\right) = \frac{q}{C}\overbrace{\frac{dq}{dt}}^{i} + L\cancel{i}\frac{di}{dt} = 0$$

$i = dq/dt$ by definition $\Rightarrow di/dt = d^2q/dt^2$

$$\Rightarrow \frac{q}{C} + L\frac{di}{dt} = 0$$

$$\Rightarrow \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\Leftrightarrow \frac{d^2q}{dt^2} = -\frac{1}{LC}q$$

$\Rightarrow$ analogous to simple harmonic motion

$$\text{cf: } \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

k = spring constant, m = mass

$\Rightarrow$ solution: $x = A\cos(\omega t + \phi)$

$\omega = \sqrt{k/m}$, A and ϕ depend on initial conditions.

$$\Longrightarrow \quad q = Q_{max}\cos(\omega t + \phi)$$

$\omega = 1/\sqrt{LC}$ natural angular frequency of the oscillation

($\omega = 2\pi f$)

f: frequency in Hz

$$i = \frac{dq}{dt} = -\omega Q_{max}\sin(\omega t + \phi)$$

at $t = 0$, $i = 0$ $\Rightarrow$ $\underline{\underline{\phi = 0}}$

$$q = Q_{max}\cos(\omega t), \quad i = -\underbrace{\omega Q_{max}}_{I_{max}}\sin \omega t$$

$$\omega = 1/\sqrt{LC}$$

$$U_{tot} = U_E + U_B$$

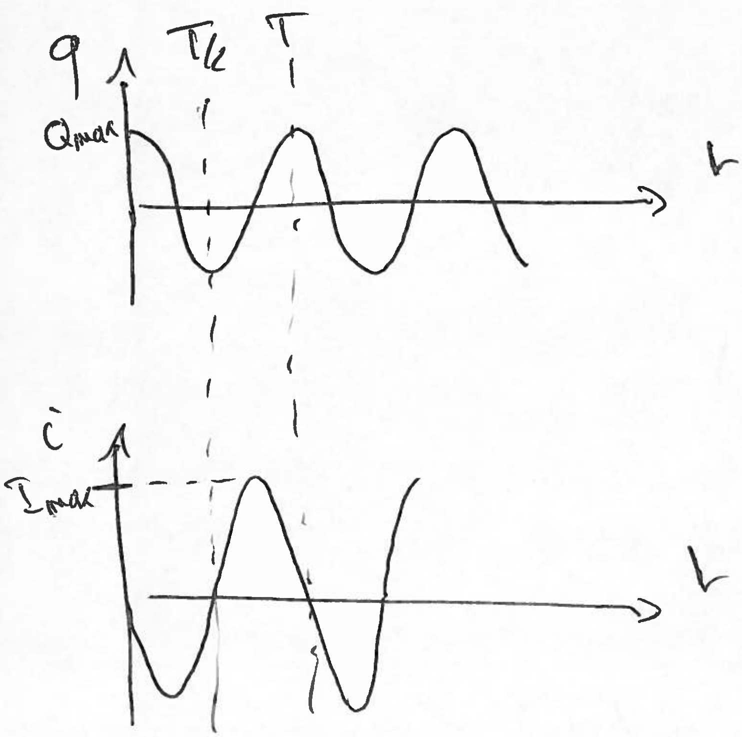
$$= \frac{Q_{max}^2}{2C} \cos^2 \omega t + \frac{1}{2} L I_{max}^2 \sin^2(\omega t)$$

$\Rightarrow$ when comparing when $q = 0$ and $i = 0$

$$\Rightarrow \left\| \frac{Q_{max}^2}{2C} = \frac{1}{2} L I_{max}^2 \right.$$

$$\Rightarrow U_{tot} = \frac{Q_{max}^2}{2C} \underbrace{\left(\cos^2 \omega t + \sin^2 \omega t\right)}_{=1} = \frac{Q_{max}^2}{2C}$$

$\Rightarrow$ ideal situation



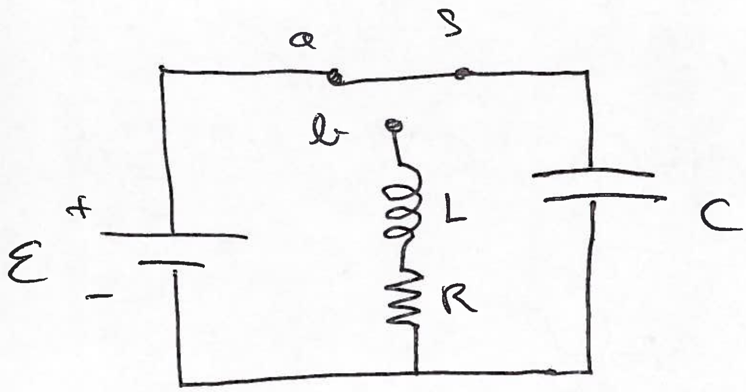
$T = 2\pi / \omega$, $\omega = 2\pi / T$

$T = 1/f$ $\qquad \omega = 2\pi / f$

The RLC circuit

↳ One need to take into account the presence of resistor in the circuit

⇒ energy transformed to internal energy in the resistor (heat)



at $t = 0$, C is fully charged $U_E = \frac{Q_{max}^2}{2C}$

⇒ switch on a

dissipation of energy in the resistor

$$\frac{dU_{tot}}{dt} = -i^2 R$$

↓

decrease of energy

$$U_{tot} = U_E + U_B = \frac{1}{2} \frac{q^2}{C} + \frac{1}{2} L i^2$$

$$\Rightarrow \frac{dU_{tot}}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2 R$$

$$i = dq/dt$$

$$Li\frac{d^2q}{dt^2} + Ri^2 + \frac{q}{C}i = 0$$

division by i

$$L\frac{d^2q}{dt^2} + Ri + \frac{q}{C} = 0$$

$$L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$\Rightarrow$ damped harmonic oscillator

cf: $m\dfrac{d^2x}{dt} + b\dfrac{dx}{dt} + kx = 0$

damping coefficient (friction, air resistance)

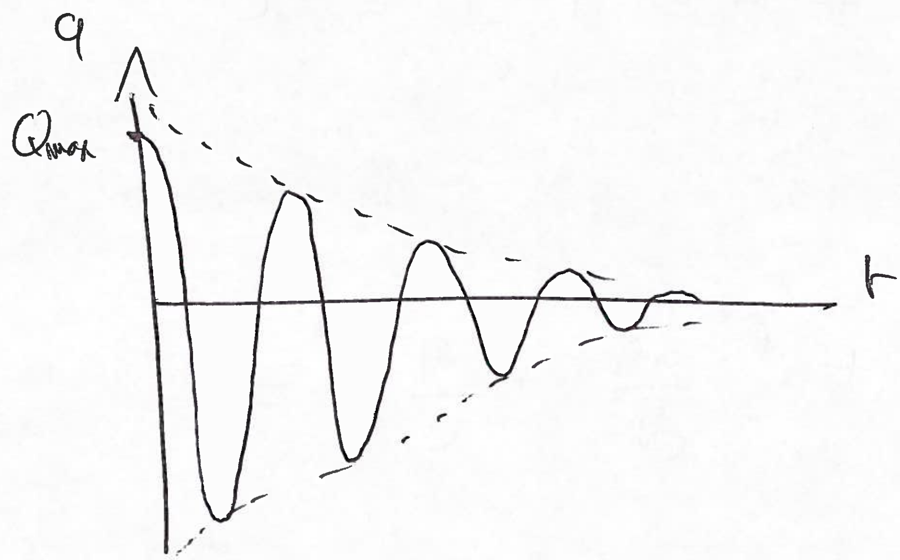
$$q = Q_{max}\, e^{-Rt/2L} \cos \omega_d t$$

with $\omega_d = \left(\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2 \right)^{1/2}$

cf block spring in a viscous medium

$$R \ll \sqrt{4L/C}$$

$$\omega_d \simeq \sqrt{1/LC}$$



<u>Critical resistance</u> $R_c = \sqrt{4L/C}$ no oscillation

critically damped

$R > R_c \Rightarrow$ overdamped

$q \leftrightarrow x$

$i \leftrightarrow v_x$

$\Delta V \leftrightarrow F_x$ (force

$R \leftrightarrow b$ viscous damping coff

$C \leftrightarrow 1/k$

$L \leftrightarrow m$