Alternating Current Circuits

Chapter 33

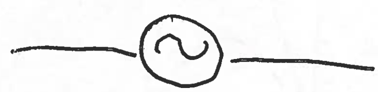
AC Sources (of generators)

$$\Delta v = \underbrace{\Delta V_{max}}_{\text{max output voltage, or voltage amplitude}} \sin \omega t$$

ω = angular frequency

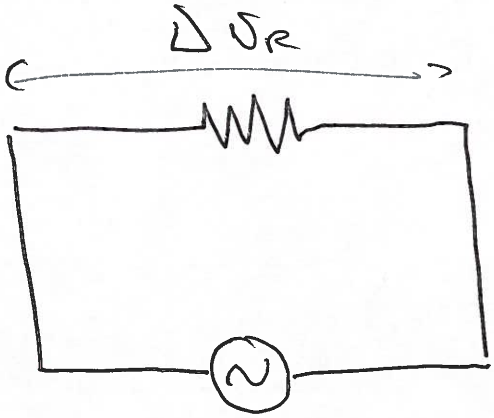
$$= 2\pi \underbrace{f}_{\text{frequency in Hz}} = 2\pi / T$$

T: Period in s.

Symbol for AC source: 

Resistor in AC circuit

(cf small letters $v, i, q, \ldots$ $\Rightarrow$ instantaneous values varying with time t)

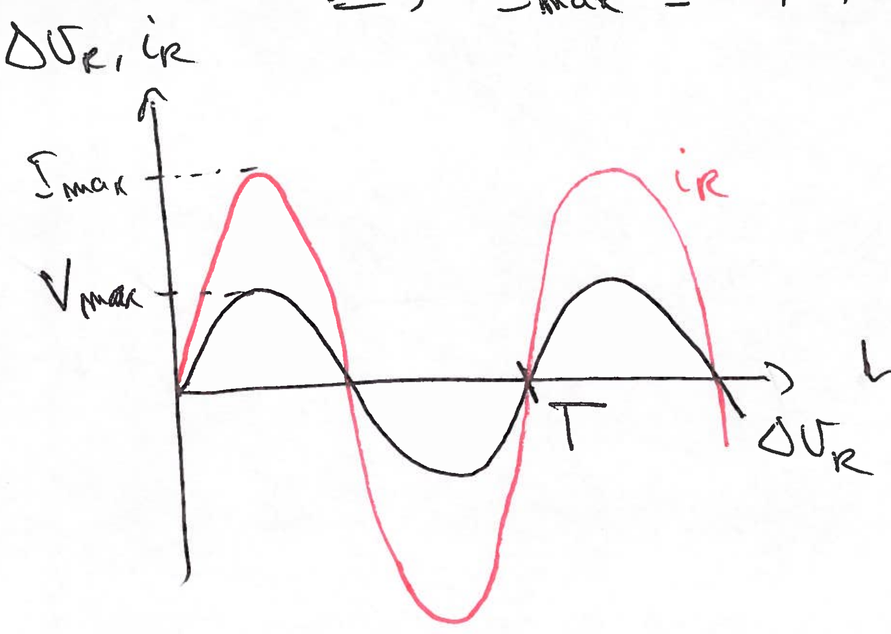


$$\Delta v = V_{max} \sin(\omega t)$$

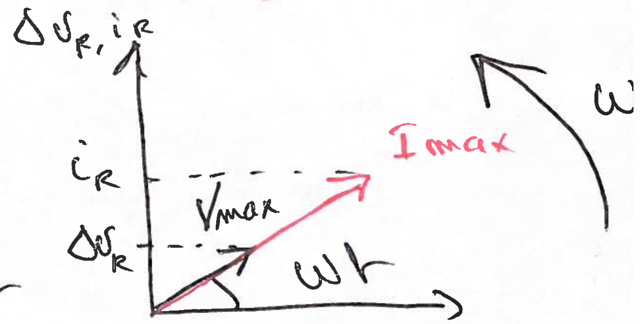
Kirchhoff's loop rule: $\Delta v - i_R R = 0$

$$\Rightarrow i_R = \frac{\Delta v}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin(\omega t)$$

$$\Rightarrow I_{max} = V_{max} / R$$



Phasor diagram



The energy transfered as internal energy in R is the same whatever is the direction of the current $\Rightarrow P = Ri^2$

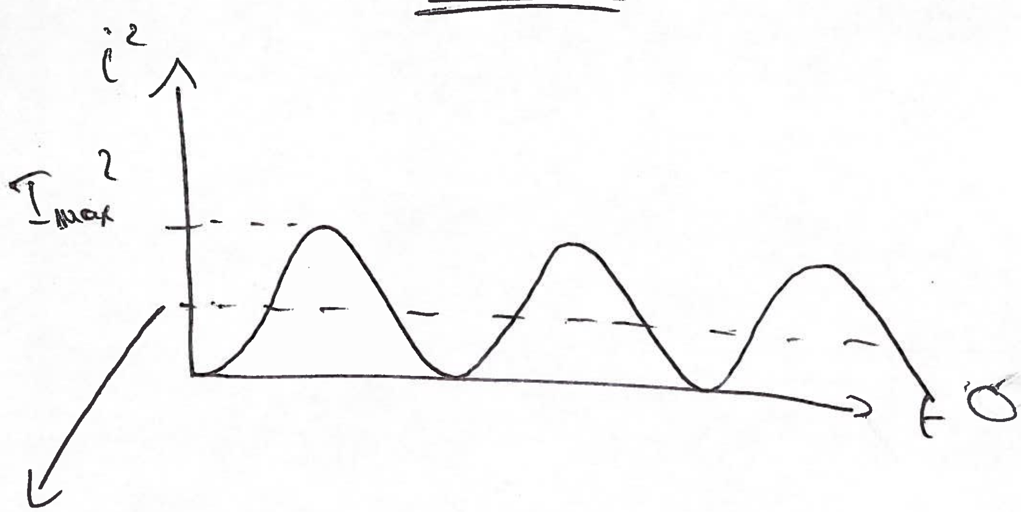
DC current of value $I_{max} \Rightarrow P = R I_{max}^2$

for AC current P is less because the current oscillates

"RMS current" $I_{rms} = \sqrt{\underbrace{(i^2)_{avg}}_{\text{average of the square of the current}}}$

"Root-Mean-Square"

Note: $\underline{\underline{i_{avg} = 0}}$



$(i^2)_{avg} = \frac{1}{2} I_{max}^2 \Rightarrow I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0{,}707\ I_{max}$

In AC $\Rightarrow$ we usually provide rms value

$$P_{avg} = R\, I_{rms}^2$$

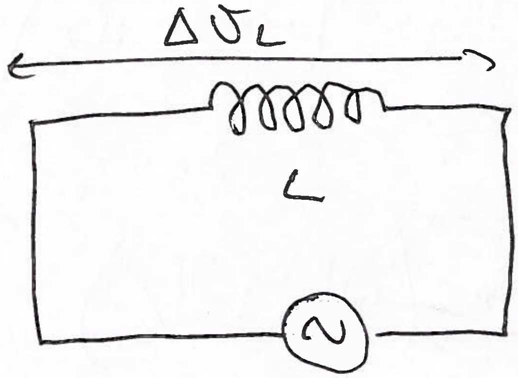
Similarly $\quad \Delta V_{rms} = \dfrac{\Delta V_{max}}{\sqrt{2}} = 0{,}707\, \Delta V_{max}$

Note an outlet providing 120V means $\Delta V_{rms} = 120V$

$\Rightarrow$ Therefore for home outlets $\Delta V_{max} = 170V$

Note 2: in AC, voltmeters and ammeters are providing rms values

Inductor in AC circuits



$$\Delta v = V_{max} \sin \omega t$$

$$\Delta v + \Delta v_L = 0$$

$$\Leftrightarrow \Delta v - L \frac{di_L}{dt} = 0$$

$$\Leftrightarrow \Delta v = L \frac{di_L}{dt} = \Delta V_{max} \sin \omega t$$

$$\Rightarrow di_L = \frac{\Delta V_{max}}{L} \sin \omega t \, dt$$

$$i_L = \frac{\Delta V_{max}}{L} \int \sin \omega t \, dt = -\frac{\Delta V_{max}}{\omega L} \cos \omega t$$

(note: we neglect the constant that depends on initial values only)

We have : $\cos \omega t = -\sin(\omega t - \pi/2)$

$$\Rightarrow -\cos \omega t = \sin(\omega t - \pi/2)$$

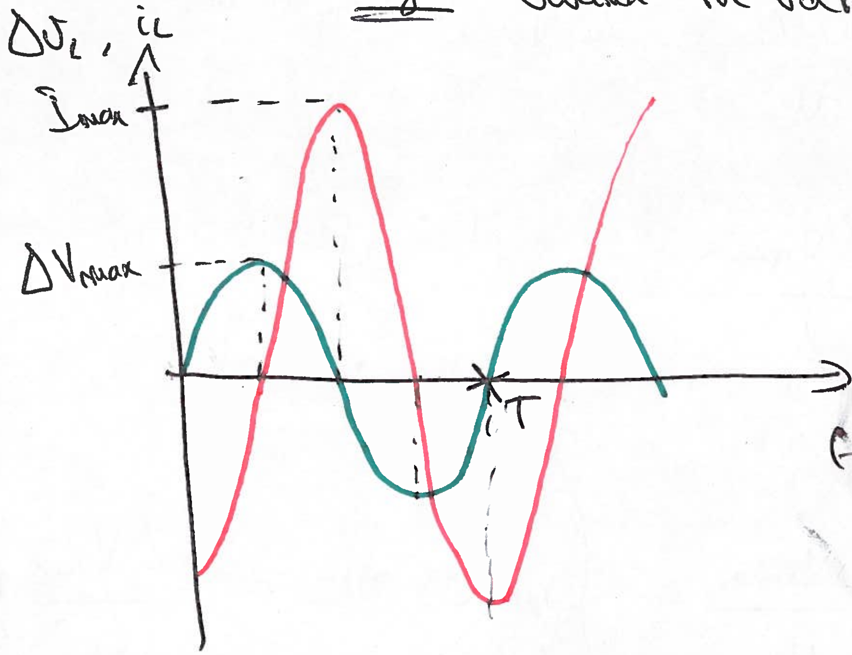
$$\Rightarrow i_L = \frac{\Delta V_{max}}{\omega L} \sin(\omega t - \pi/2)$$

I_{max}

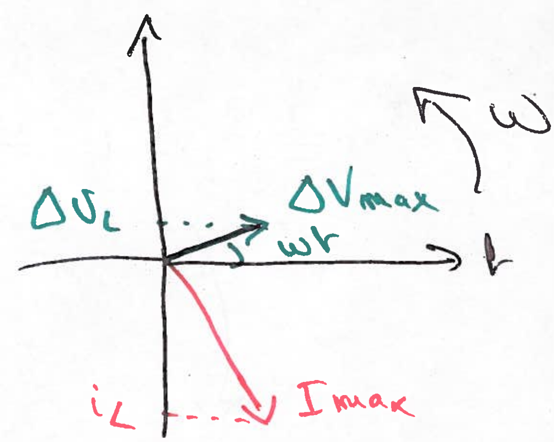
i_L and Δv_L are out of phase by

$\pi/2$ rad $= 90°$

the current lags behind the voltage



Phasor diagram



when i_L is max, no change $\Rightarrow \Delta v_L = 0$

when i_L is 0, max rate change $\Rightarrow \Delta v_L = \max$

$$I_{max} = \frac{\Delta V_{max}}{\omega L}$$

if in DC $\Rightarrow I = \frac{\Delta V}{R}$

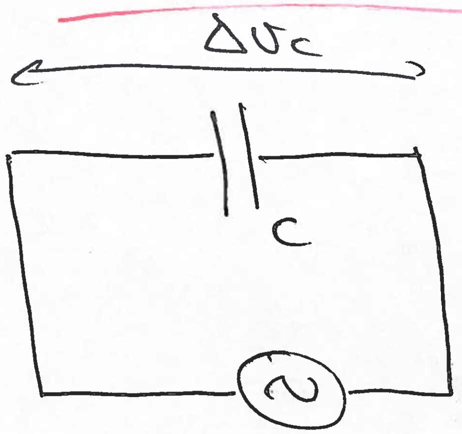
$\Rightarrow$ $X_L \equiv \omega L$: Inductive reactance

$$I_{max} = \frac{\Delta V_{max}}{X_L} \qquad I_{rms} = \frac{\Delta V_{rms}}{X_L}$$

When f (or ω) $\nearrow$ $\Rightarrow X_L \nearrow \Rightarrow$ current $\searrow$

(of Faraday law $\Rightarrow$ $\nearrow$ rate of change $\Rightarrow$ $\nearrow$ back emf)

Capacitors in AC circuits



$$\Delta V = V_{max} \sin \omega t$$

Kirchhoff's loop rule

$$\Delta V + \Delta V_c = 0$$

$$\Delta V - \frac{q}{C} = 0$$

$$\Rightarrow \quad q = C \Delta V_{max} \sin \omega t$$

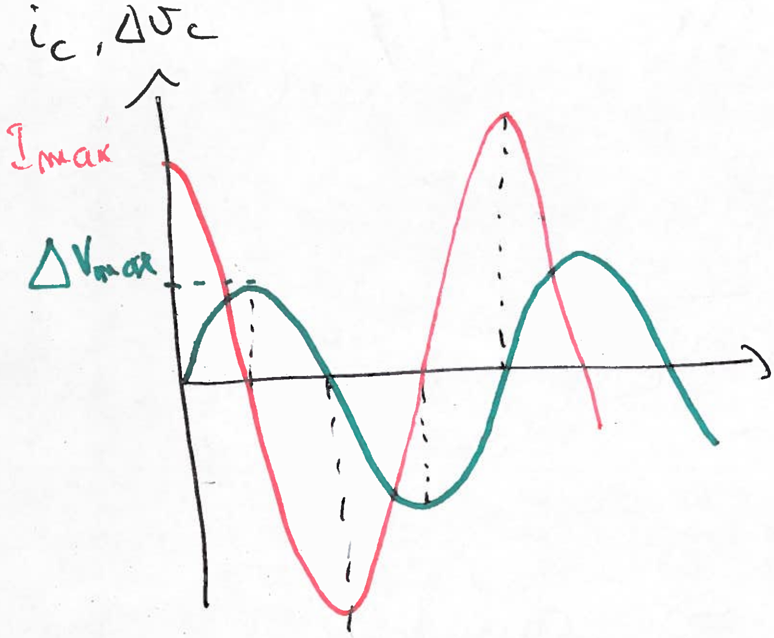
$$i_c = \frac{dq}{dt} = \omega C \Delta V_{max} \cos \omega t$$

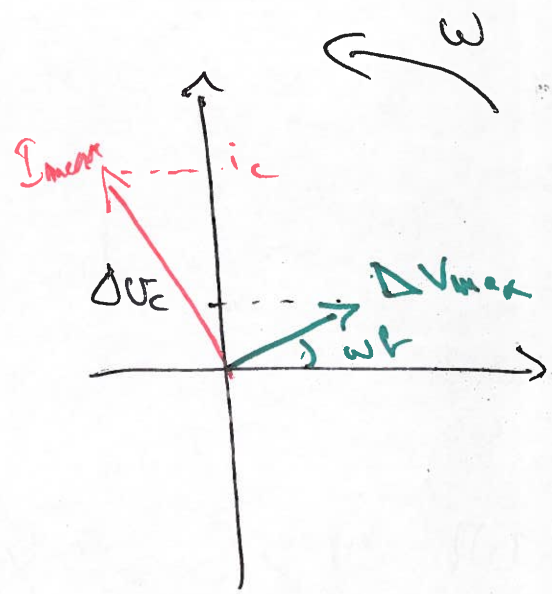
We have $\cos \omega t = \sin\left(\omega t + \pi/2\right)$

$$i_C = \omega C \Delta V_{max} \sin(\omega t + \pi/2)$$

i_C and Δv_C are out of phase

i_C precedes Δv_C





at $\Delta v = \Delta V_{max}$ ⇒ C fully charged ⇒ $\underline{i_C = 0}$

when $i_C = max$ ⇒ Δv_C is fully discharge and starts changing $\underline{\text{polarity}}$

$$I_{max} = \omega C \, \Delta V_{max} = \frac{\Delta V_{max}}{(1/\omega C)}$$

cf $I = \dfrac{\Delta V}{R}$

Capacitive Reactance $\quad X_C \equiv 1/\omega C$

$$\Rightarrow I_{max} = \frac{\Delta V_{max}}{X_C}$$

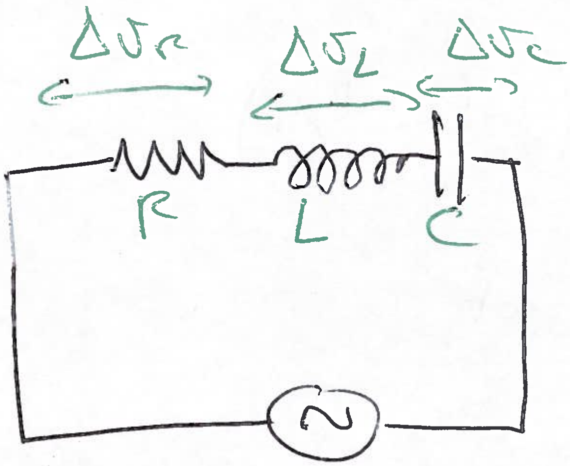
When $f \nearrow \Rightarrow X_C \downarrow \Rightarrow$ current $\nearrow$

When $f \searrow \Rightarrow X_C \nearrow \Rightarrow$ when f close to 0

$\Rightarrow$ open circuit

(cf steady state)

The RLC serie circuit



$$\Delta V = \Delta V_{max} \sin \omega t$$

$$\Rightarrow \quad i = I_{max} \sin(\omega t - \underbrace{\phi}_{\text{phase angle}})$$

3 elements in <u>serie</u> $\Rightarrow$ the current should be the <u>same</u> everywhere

* ΔV_R is in phase with current

$$\Rightarrow \Delta V_R = R\, I_{max} \sin(\omega t)$$
$$= \Delta V_R \sin(\omega t)$$

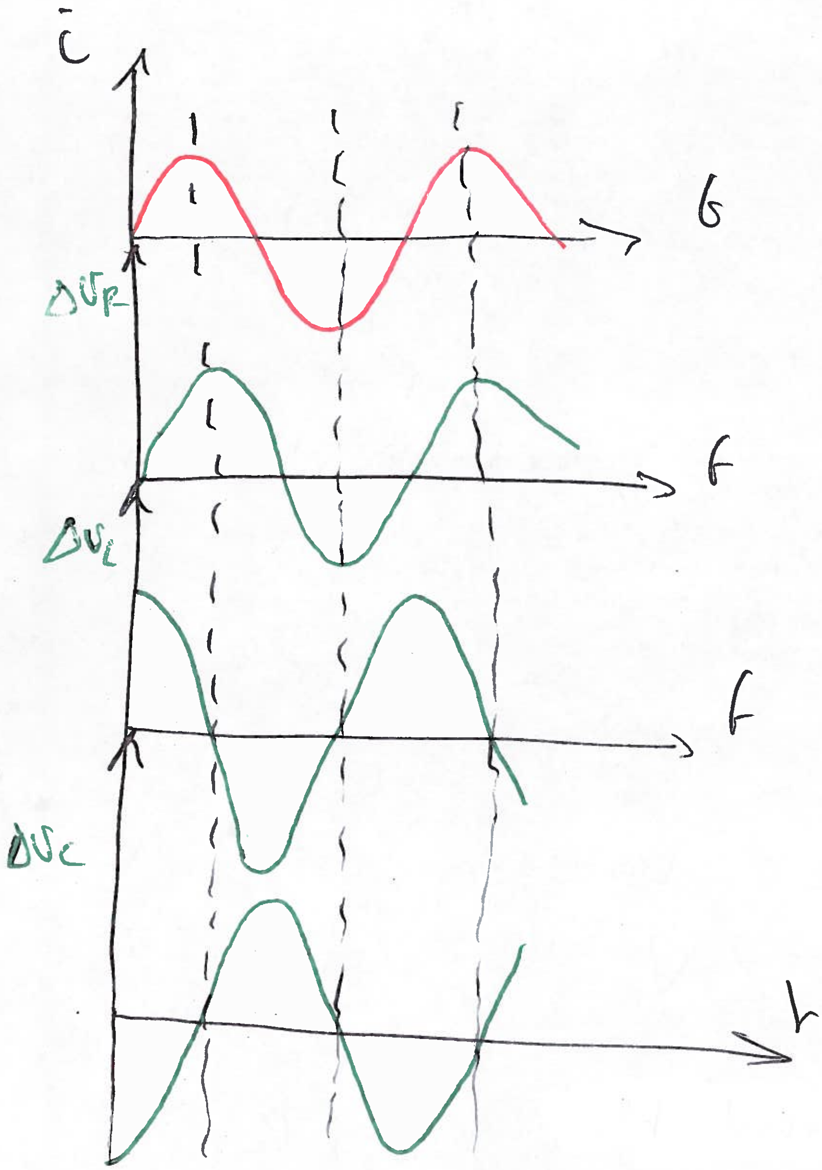
* ΔV_L <u>leads</u> the current by $90°$ $(\pi/2)$

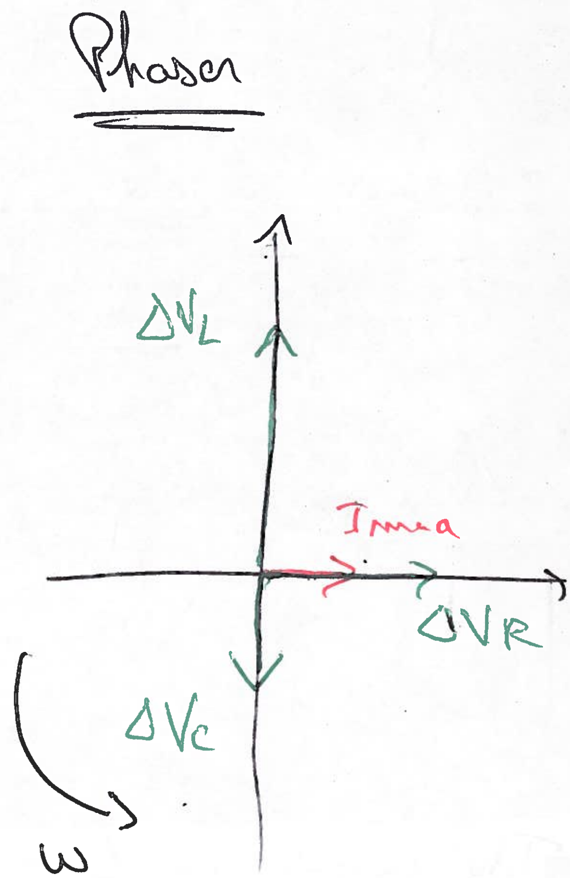
$$\Rightarrow \Delta V_L = I_{max} X_L \sin(\omega t + \pi/2)$$
$$= \Delta V_L \cos(\omega t)$$

* Δv_C lags behind the current by $90°$ $(\pi/2)$

$$\Delta v_C = I_{max} X_C \sin(\omega t - \pi/2)$$

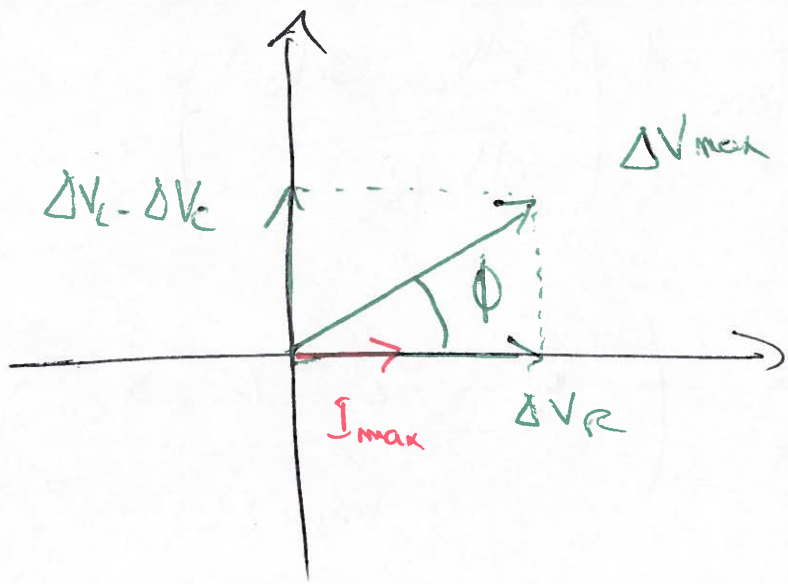
$$= -\Delta V_C \cos \omega t$$





(at an instant at which $i = 0$)

$\Longleftrightarrow$



$\Longrightarrow$ Vectorial summation $\qquad \Delta V = \Delta V_R + \Delta V_L + \Delta V_C$

$$\Longrightarrow \Delta V_{max} = \sqrt{\Delta V_R^2 + (\Delta V_L - \Delta V_C)^2}$$

$$= \sqrt{(I_{max} R)^2 + (I_{max} X_L - I_{max} X_C)^2}$$

$$= I_{max} \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \quad \text{in ohm}$$

$$I_{max} = \frac{\Delta V_{max}}{Z} \Longleftrightarrow I_{rms} = \frac{\Delta V_{rms}}{Z}$$

$$\phi = \tan^{-1}\left(\frac{\Delta V_L - \Delta V_C'}{\Delta V'_R}\right)$$

$$= \tan^{-1}\left(\frac{I_{max} X_L - I_{max} X_C}{I_{max} R}\right)$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

$X_L > X_C$, at higher frequencies $\Rightarrow \phi > 0$

$\Rightarrow$ the current <u>lags</u> behind the voltage

$\Rightarrow$ the circuit is more inductive than capacitive

$X_L < X_C$, at lower frequencies $\Rightarrow \phi < 0$

$\Rightarrow$ the current <u>leads</u> the applied voltage

$\Rightarrow$ the circuit is more capacitive than inductive

$X_L = X_C \Rightarrow$ circuit purely <u>resistive</u>

Power in an AC circuit

Instantaneous power

$$P = i\,\Delta v = I_{max} \sin(\omega t - \phi)\, \Delta V_{max} \sin(\omega t)$$

$$P = \underline{I_{max} \Delta V_{max}} \sin(\omega t) \sin(\omega t - \phi)$$

↳ average power over one or more cycle

We are using: $\sin(\omega t - \phi) = \sin\omega t \cos\phi - \cos\omega t \sin\phi$

$$\Rightarrow P = I_{max} \Delta V_{max} \sin^2\omega t \cos\phi - I_{max} \Delta V_{max} \sin\omega t \cos\omega t \sin\phi$$

Average of $\sin^2 \omega t$ over 1 or more cycle $\Rightarrow \underline{1/2}$

Average of $\sin\omega t \cos\omega t$ over one or more cycle

$$\Rightarrow \sin\omega t \cos\omega t = \frac{1}{2} \sin 2\omega t \Rightarrow \text{average} = \underline{0}$$

$$\Rightarrow \boxed{\begin{aligned} P_{avg} &= \frac{1}{2} I_{max} \Delta V_{max} \cos\phi \\ P_{avg} &= I_{rms} \Delta V_{rms} \cos\phi \end{aligned}}$$

power factor

From phasor diagram

$$\cos\phi = \Delta V_R / \Delta V_{max} = R\, I_{max} / \Delta V_{max}$$

$$\Rightarrow \cos\phi = R / Z$$

$$\Rightarrow P_{avg} = I_{rms}\, \Delta V_{rms} \cos\phi$$

$$= I_{rms}\, \Delta V_{rms} \left(R/Z \right)$$

$$= I_{rms} \underbrace{\left(\frac{\Delta V_{rms}}{Z} \right)}_{I_{rms}} R = R\, I_{rms}^2$$

$$P_{avg} = R\, I_{rms}^2$$

↳ average power converted into internal energy in the resistor, as in DC circuit

When $\phi = 0$, $\cos\phi = 1 \Rightarrow P_{avg} = I_{rms}\, \Delta V_{rms}$.

$\Rightarrow$ no power losses associated with pure capacitor and pure inductor cf: the capacitor charges and discharges twice over each cycle $\Rightarrow$ average energy $= 0$

Idem for the inductor

Resonance in a series RLC circuit

Circuit in resonance $\Rightarrow$ the driving frequency is such that the rms current is max

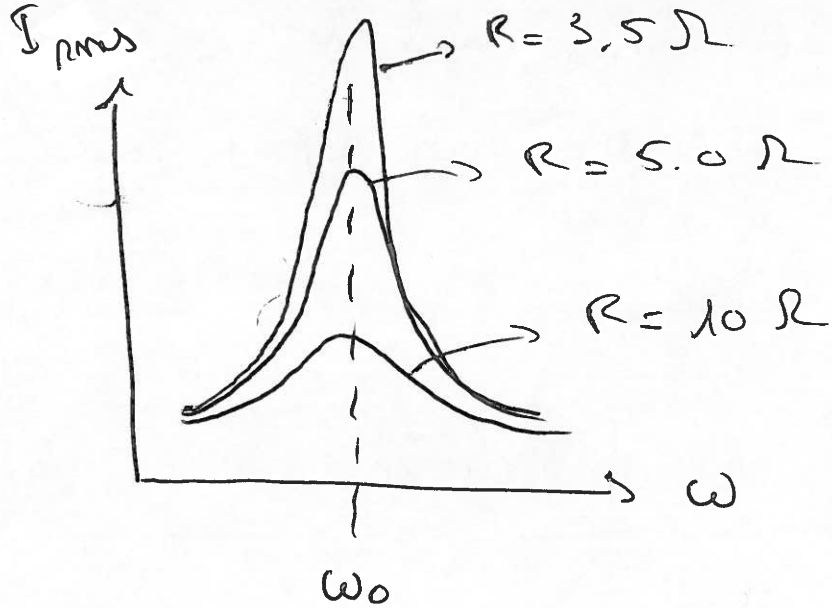
$$I_{rms} = \frac{\Delta V_{rms}}{Z} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

I_{rms} maximum at ω_0 at which $X_L = X_C$
$(X_L - X_C = 0)$

$$X_L = X_C \Leftrightarrow \omega_0 L = \frac{1}{\omega_0 C}$$

$$\Leftrightarrow \quad \omega_0 = 1/\sqrt{LC}$$

Resonance frequency = natural oscillation of an LC circuit



$$P_{avg} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R$$

$$= \frac{(\Delta V_{rms})^2 R}{R^2 + (X_L - X_C)^2}$$

$X_L = \omega L$, $X_C = 1/\omega C$, $\omega_0^2 = 1/LC$

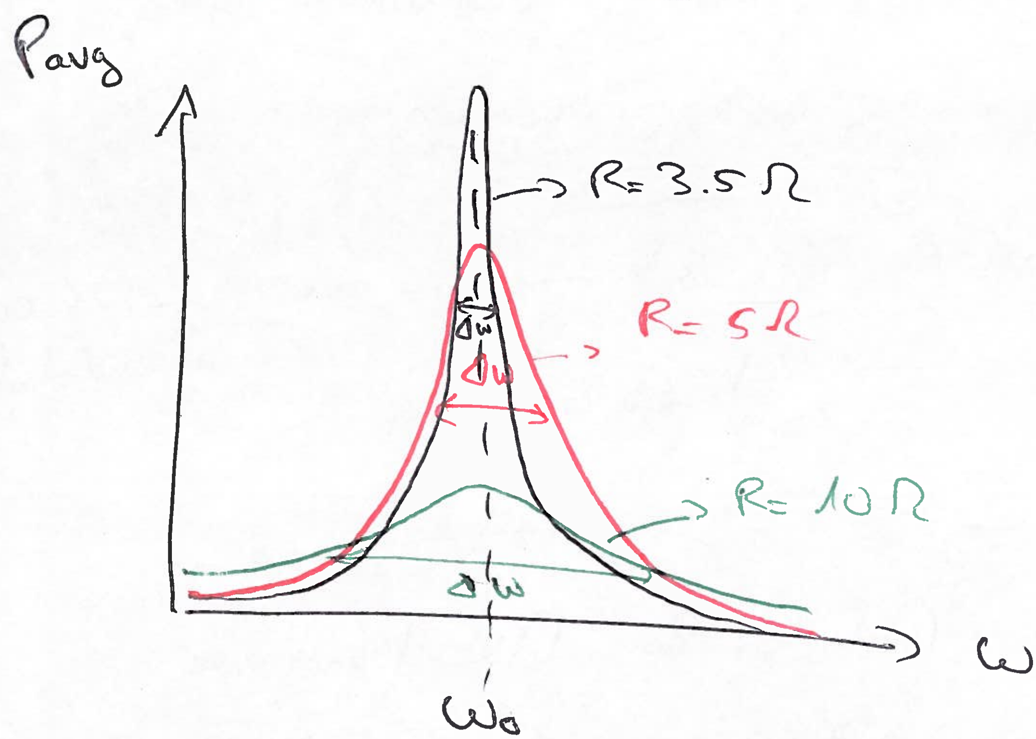
$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$= \frac{L^2}{\omega^2}\left(\omega^2 - \omega_0^2\right)^2$$

$$\Rightarrow P_{avg} = \frac{(\Delta V_{rms})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2-\omega_0^2)^2}$$

$\Rightarrow$ P_{avg} is maximum when $\omega = \omega_0$

and has the value $(\Delta V_{rms})^2/R$



$\Delta\omega \Rightarrow$ half power point

$= R/L$

Quality factor $\quad Q = \dfrac{\omega_0}{\Delta\omega} = \dfrac{\omega_0 L}{R}$

$\Rightarrow$ eg. Radio. receiving circuit

The radio is tuned by varying a capacitor, which changes ω_0 $\Rightarrow$ signal passes through amplifier and speakers

⇒ high-Q circuits are needed to eliminate unwanted signals (from other radio station)

Transformer and Power transmission

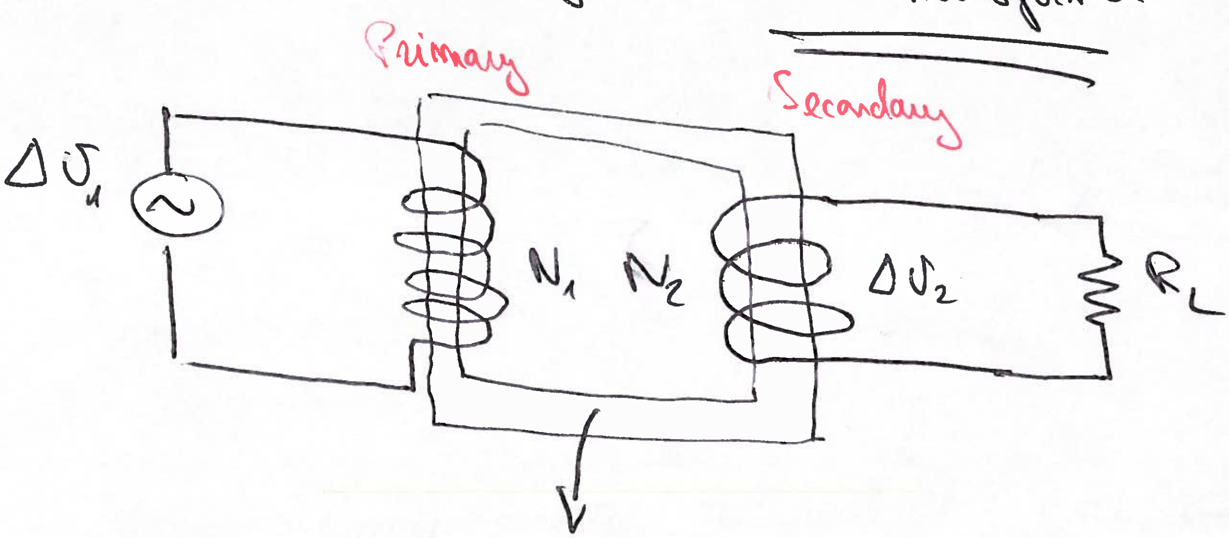
To minimize RI^2 loss in transmission lines

⇒ high voltage transmission

Usually 350 or 765 kV → 20.000 V → 4.000 V (residential area)

→ 120 V
240 V at customer's site

Role of the AC transformer



iron core to ↗ magnetic flux through the coils ⇒ all magnetic field lines inside the core ⇒ laminated core to ↘ eddy currents

$$\Delta V_1 = -N_1 \frac{d\phi_B}{dt}$$

same $d\phi_B/dt$ because of the iron core

$$\Delta V_2 = -N_2 \frac{d\phi_B}{dt}$$

$$\Rightarrow \boxed{\Delta V_2 = \frac{N_2}{N_1} \Delta V_1}$$

$N_2 > N_1$: step up transformer

$N_1 > N_2$: step down transformer.

Ideal transformer $\Rightarrow$ no loss in power through transmission

$$I_{1\,rms}\, \Delta V_{1\,rms} = I_{2\,rms}\, \Delta V_{2\,rms}$$

$$\Rightarrow I_{2\,rms} = \frac{\Delta V_{2\,rms}}{R_L} = \frac{I_{1\,rms}\, \Delta V_{1\,rms}}{\Delta V_{2\,rms}}$$

$$\Leftrightarrow \frac{(\Delta V_{2\,rms})^2}{R_L} = I_{1\,rms}\, \Delta V_{1\,rms}$$

$$\Leftrightarrow \quad \frac{(N_2/N_1)^2 \, \Delta V_{1\,rms}^{\cancel{2}}}{R_L} = I_{1\,rms} \, \cancel{\Delta V_{1\,rms}}$$

$$\Leftrightarrow \quad \Delta V_{1\,rms} = \underbrace{\left(\frac{N_1}{N_2}\right)^2 R_L}_{R_{eq}} \quad I_{1\,rms}$$

R_{eq} ⇒ equivalent resistance view from the primary circuit

$$R_{eq} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

⇒ a transformer is matching the resistance between the primary circuit and the load

⇒ max power transfer

ex: A transformer to connect a $1\,k\Omega$ output of the audio amp to a $8\,\Omega$ speaker.

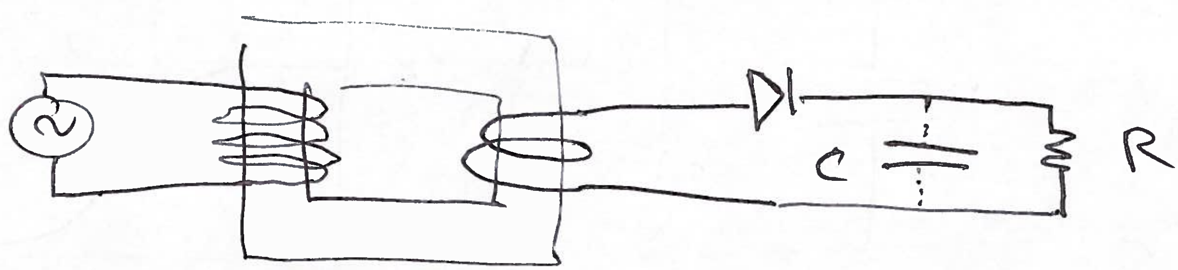
⇒ Impedance matching

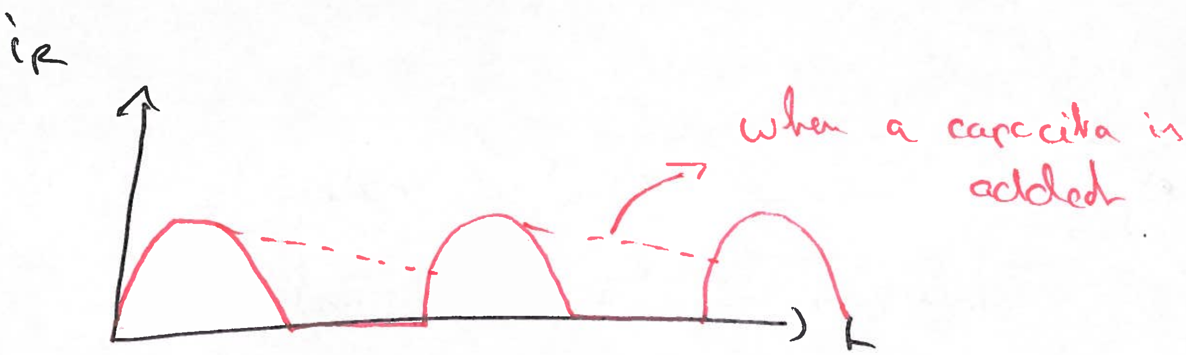
Rectifier and filters

To convert AC to DC for most devices (radios, laptop, ...) => Rectifier

Use of a diode

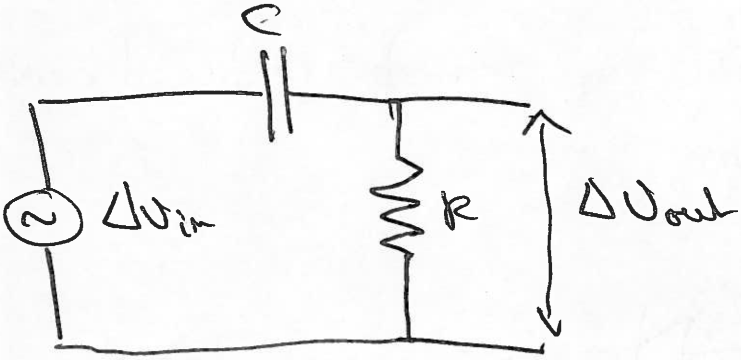
low resistance in the direction of the arrow, high resistance in the other direction.

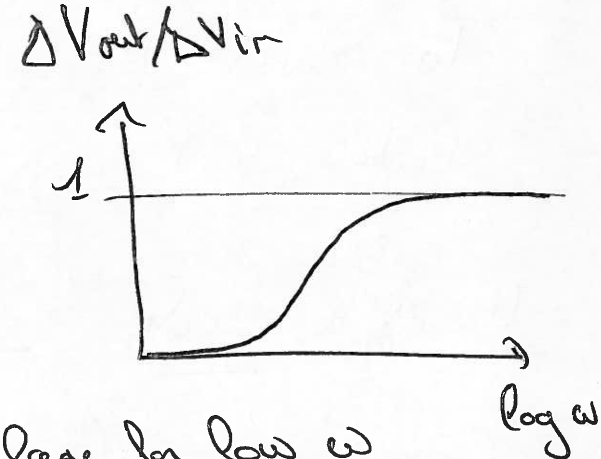




One need to remove small 60 Hz oscillat° (ripple)

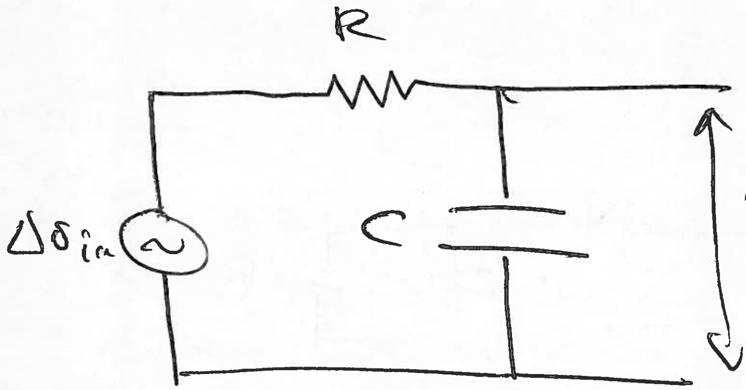
RC high pass filter

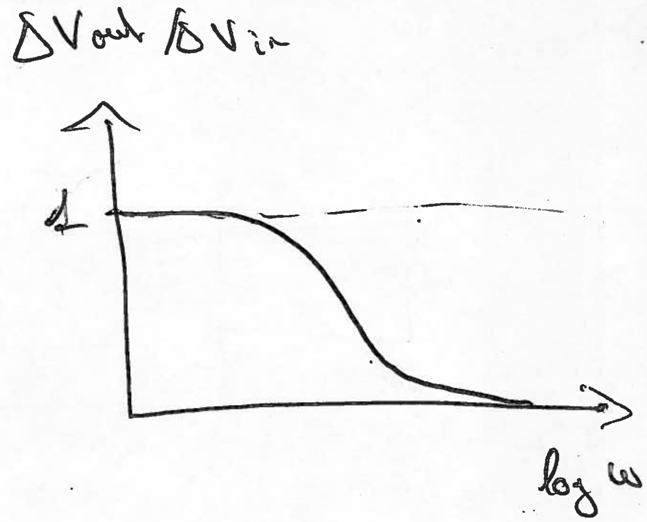




cf: $X_C = 1/C\omega$, large for low ω

RC low pass filter





cf: low pass ⇒ "woofer" speaker

high pass ⇒ "tweeter" speaker