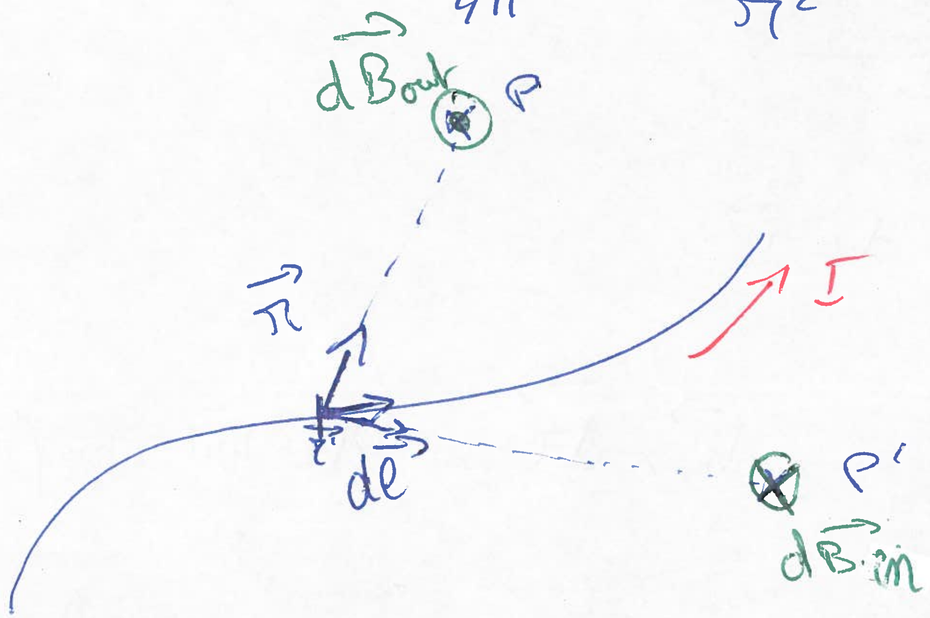
Sources of magnetic fields

(Chapter 30)

Biot - Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \vec{r}}{r^2}$$

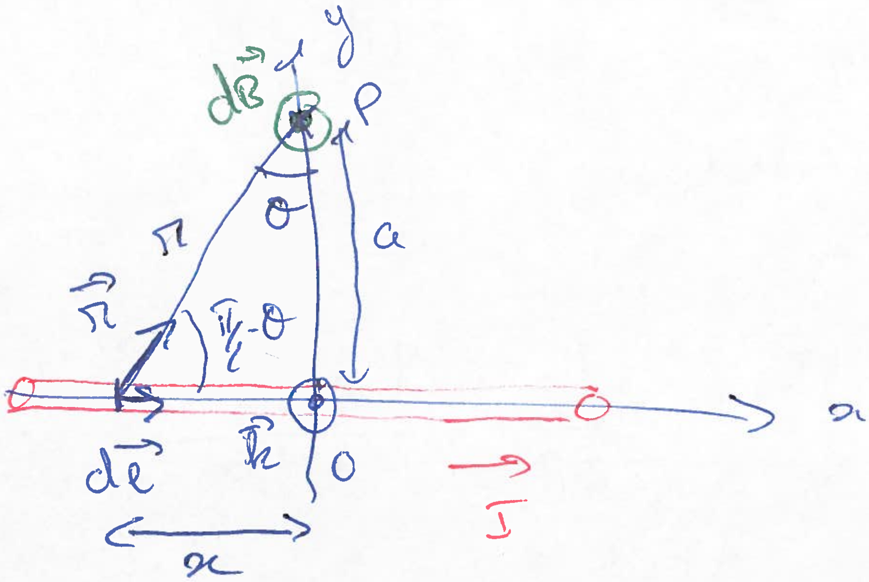


$$\mu_0 = 4\pi \cdot 10^{-7} \ T \cdot m/A$$

permeability of free space

$$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^2}$$

Magnetic field generated by a thin, straight conductor



$$\vec{dl} = dx\,\vec{i}$$

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{\vec{dl}\times\vec{r}}{r^2} \qquad \vec{dl}\times\vec{dr} = dx\,\underbrace{\|\vec{r}\|}_{=1}\sin(\pi/2-\theta)\,\vec{k}$$

$$= dx\cos\theta\,\vec{k}$$

$$\Rightarrow \vec{dB} = \frac{\mu_0 I}{4\pi}\frac{dx\cos\theta}{r^2}\,\vec{k}$$

$$r = a/\cos\theta \quad , \quad x = -a\tan\theta$$

(x is <0 on $\vec{i}$)

$$\frac{dx}{d\theta} = -\frac{a}{\cos^2\theta} = -a\sec^2\theta$$

$$dx = -\frac{a\,d\theta}{\cos^2\theta}$$

$$\Rightarrow \|\vec{dB}\| = -\frac{\mu_0 I}{4\pi}\left(\frac{\cancel{a}\,d\theta}{\cancel{\cos^2\theta}}\right)\left(\frac{\cancel{\cos^2\theta}}{a^{\cancel{2}}}\right)\cos\theta$$

$$= -\frac{\mu_0 I}{4\pi a}\cos\theta\,d\theta$$

$$B = -\int_{\theta_1}^{\theta_2}\frac{\mu_0 I}{4\pi a}\cos\theta\,d\theta$$

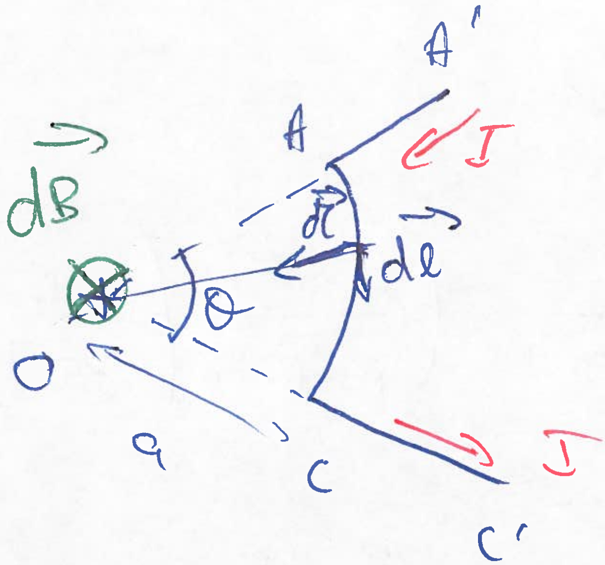
$$\vec{B} = \frac{\mu_0 I}{4\pi a}\left(\sin\theta_1 - \sin\theta_2\right)\vec{k}$$

$\theta_1 = \pi/2 \quad \theta_2 = -\pi/2 \Rightarrow \sin\theta_1 - \sin\theta_2 = 1-(-1)$

$= 2$

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\vec{k}$$

Magnetic field generated by a curved wire segment



Along A, A' and CC'

$$\vec{dl} \parallel \vec{r} \Rightarrow \underline{\underline{\vec{dB} = \vec{0}}}$$

on AC: $\vec{dl} \perp \vec{r}$

$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{dl \, \overbrace{\|\vec{r}\|}^{1}}{a^2} \, (-\vec{k})$$

$$dB = \|\vec{dB}\| = \frac{\mu_0 I}{4\pi} \frac{dl}{a^2}$$

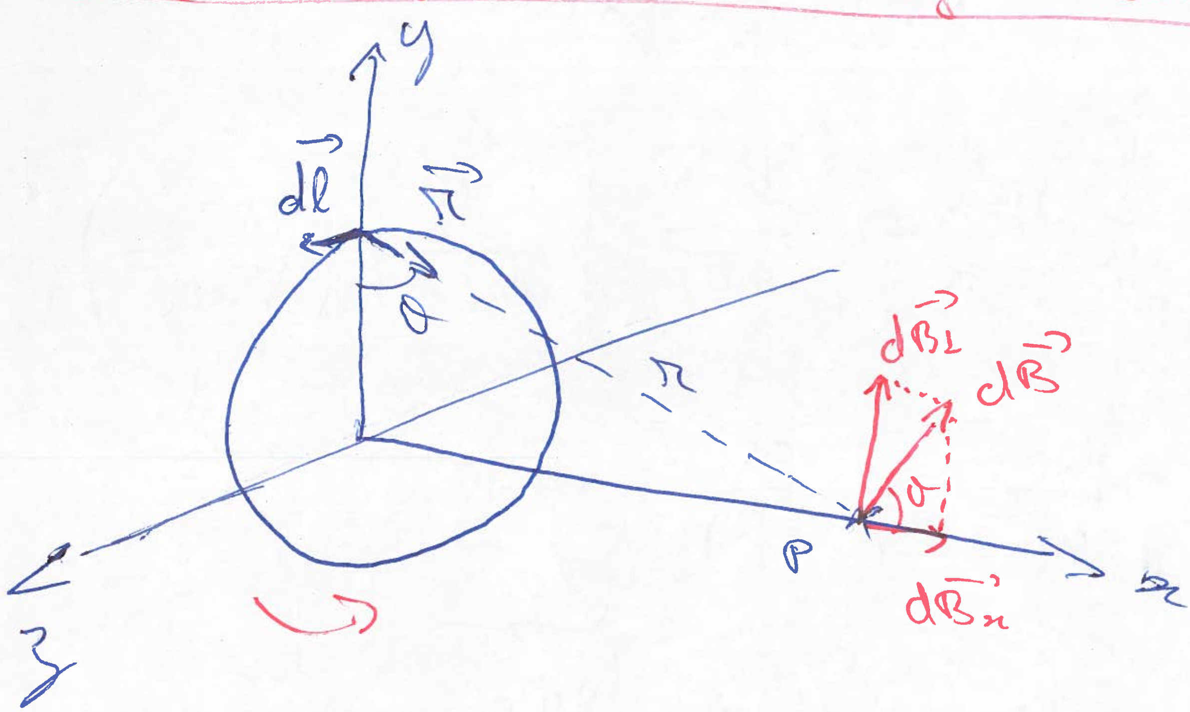
$$B = \|\vec{B}\| = \frac{\mu_0 I}{4\pi a^2} \underbrace{\int dl}_{= l = a\theta}$$

$$\|\vec{B}\| = \frac{\mu_0 I}{4\pi a^2} a\theta \quad (-\vec{k})$$

$\Rightarrow$ for a ring $\Rightarrow \theta = 2\pi$

$$\Rightarrow \|\vec{B}\| = \frac{\mu_0 I}{4\pi a^2} a \, 2\pi = \frac{\mu_0 I}{2a}$$

Magnetic field on the axis of a circular loop



By symmetry: $\vec{B}_\perp = \int d\vec{B}_\perp = \vec{0}$

$\Rightarrow \vec{B} = B_x \vec{i}$

$$dB = \frac{\mu_0 I}{4\pi} \frac{|\vec{dl} \times \vec{r}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{dl}{(a^2+x^2)}$$

$$\| \; dl \perp \vec{r} \quad \forall \vec{dl}$$

$$dB_x = dB \cos\theta = \frac{\mu_0 I}{4\pi} \frac{dl}{(a^2+x^2)} \cos\theta$$

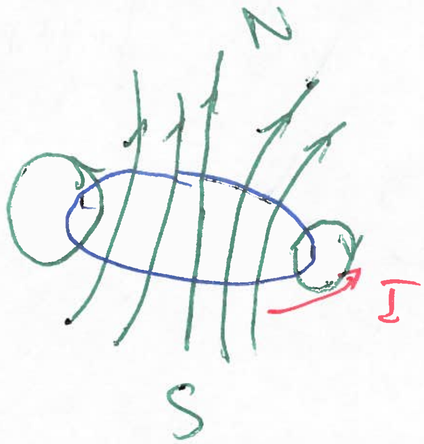
$$\cos\theta = \frac{a}{\sqrt{r^2}} = \frac{a}{\sqrt{a^2+x^2}}$$

$$B_x = \oint dB_x = \frac{\mu_0 I}{4\pi} \oint \frac{dl}{a^2+x^2} \left(\frac{a}{\sqrt{a^2+x^2}} \right)$$

$$= \frac{\mu_0 I}{4\pi} \frac{a}{(a^2+x^2)^{3/2}} \underbrace{\oint dl}_{2\pi a}$$

$$\| \; \vec{B} = \frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}} \vec{i}$$

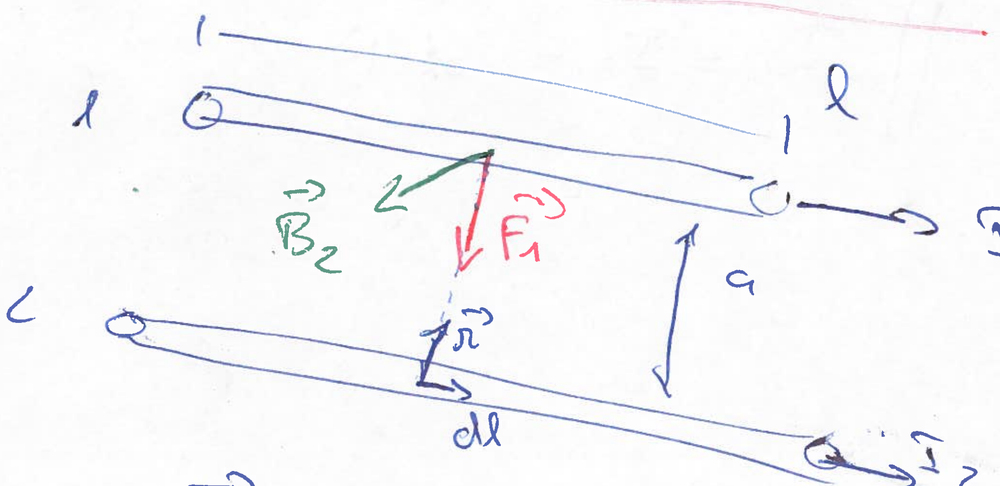
$\underline{x = 0} \Rightarrow \quad \vec{B} = \dfrac{\mu_0 I}{2a} \vec{i}$



$\underline{x \gg a} \quad \vec{B} \sim \dfrac{\mu_0 I a^2}{2 x^3} \vec{i}$

(cf similar to an electric dipole: $E = \dfrac{k_e p}{y^3}$)

Magnetic force between 2 // conductors



$\vec{B}_2 = \dfrac{\mu_0 I}{2\pi a} \vec{h}$

$\vec{F}_1 = I_1 \vec{l} \times \vec{B}_2$

$$\|\vec{F_1}\| = I_1 \, l \, B_2 = I_1 \, l \left(\frac{\mu_0 I_2}{2\pi a} \right)$$

$$\frac{F_1}{l} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

$$\|\vec{F_1}\| = \|\vec{F_2}\|$$

I_1 and I_2 in same direction $\Rightarrow$ attraction

I_1 and I_2 in opposite directions $\Rightarrow$ repulsion

Definition of the Ampere

When F_1/l = force / unit length $= 2 \times 10^{-7}$ N/m

for 2 wires separated by $a = 1$ m $\Rightarrow$

$$\| \quad I_1 = I_2 = 1 \text{ A}$$

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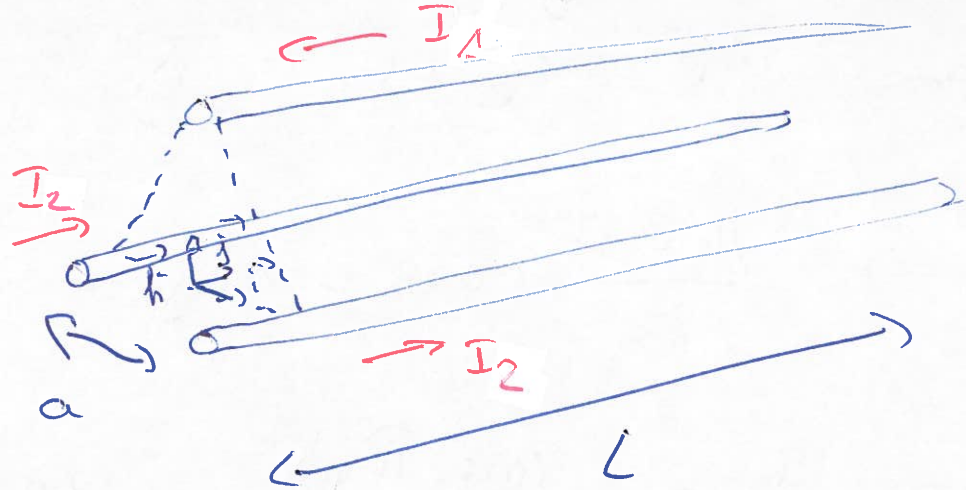
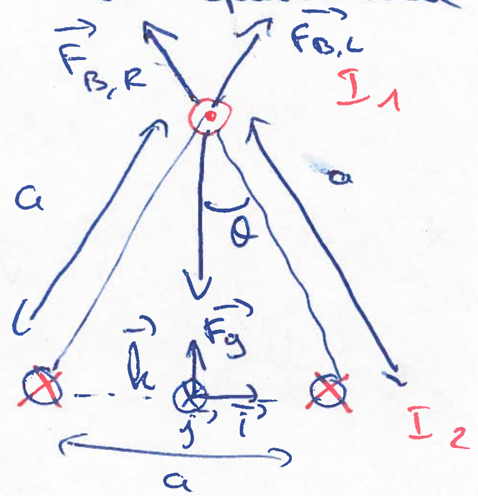


$a = 1.00$ cm $\quad L = 10.0$ m

mass of the wire $m = 400$ g $\quad I_1 = 100$ A

What should be I_2 so that the triangle would be equilateral?



for a triangle equilateral θ should be $30°$

$$\vec{F}_B = 2\left(\frac{\mu_0 I_1 I_2}{2\pi a} L\right)\cos\theta\,\vec{k} = \frac{\mu_0 I_1 I_2}{\pi a}\cos\theta\,\vec{k}$$

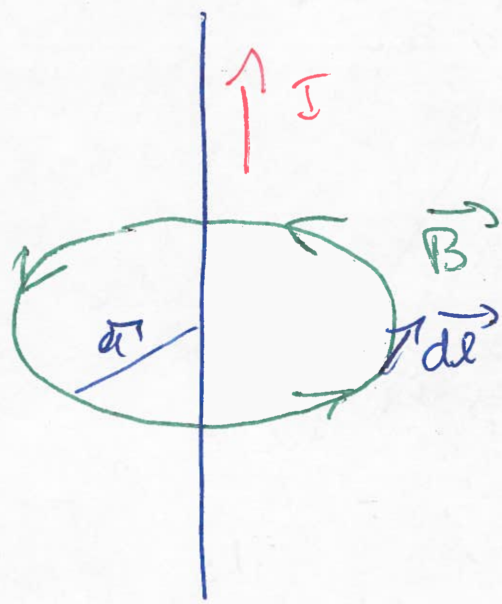
Gravitational force : $\vec{F}_g = -mg\vec{k}$

Equilibrium : $\sum \vec{F} = \vec{F_B} + \vec{F_g} = \vec{0}$

$$\Rightarrow \quad \frac{\mu_0 I_1 I_2}{\pi a} l \cos\theta - mg = 0$$

$$\Rightarrow \quad I_2 = \frac{mg\pi a}{\mu_0 I_1 l \cos\theta} = 113\ A \quad (\theta = 30^\circ)$$

Ampère's law



$$B = \frac{\mu_0 I}{2\pi r}$$

$$\vec{B} \cdot \vec{dl} = B\, dl$$

(circulation along the path according to the right thumb rule)

$$\oint_C \vec{B}\cdot d\vec{l} = \oint B\,dl = \underbrace{B}_{\frac{\mu_0 I}{2\pi r}} \underbrace{\oint dl}_{2\pi r}$$

$$\Rightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

<u>Generalization</u>

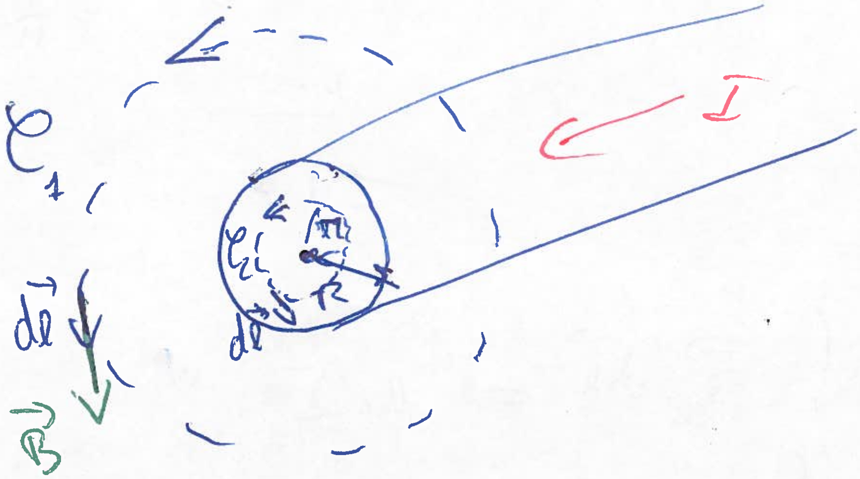
<u>Ampère's law</u>

Through <u>any</u> close path

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

<u>total</u> steady state current passing through any surface bounded by the closed path.

Magnetic field created by a long current carrying wire



$r > R$

$$\oint_{\mathcal{C}_1} \vec{B} \cdot \vec{dl} = B \cdot (2\pi r) = \mu_0 I$$

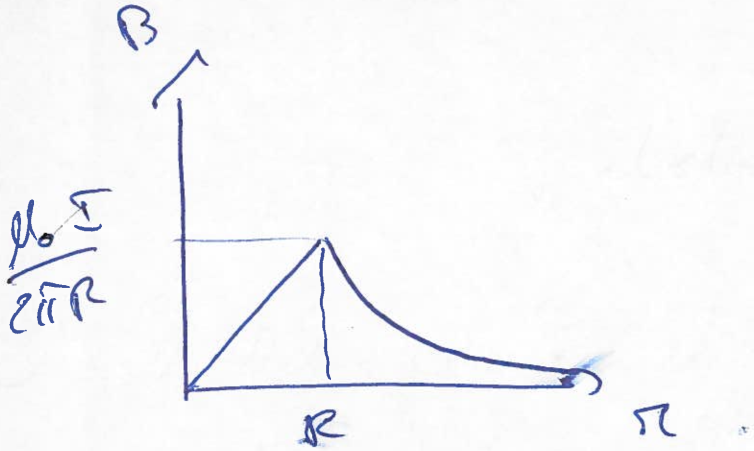
$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

$r < R$

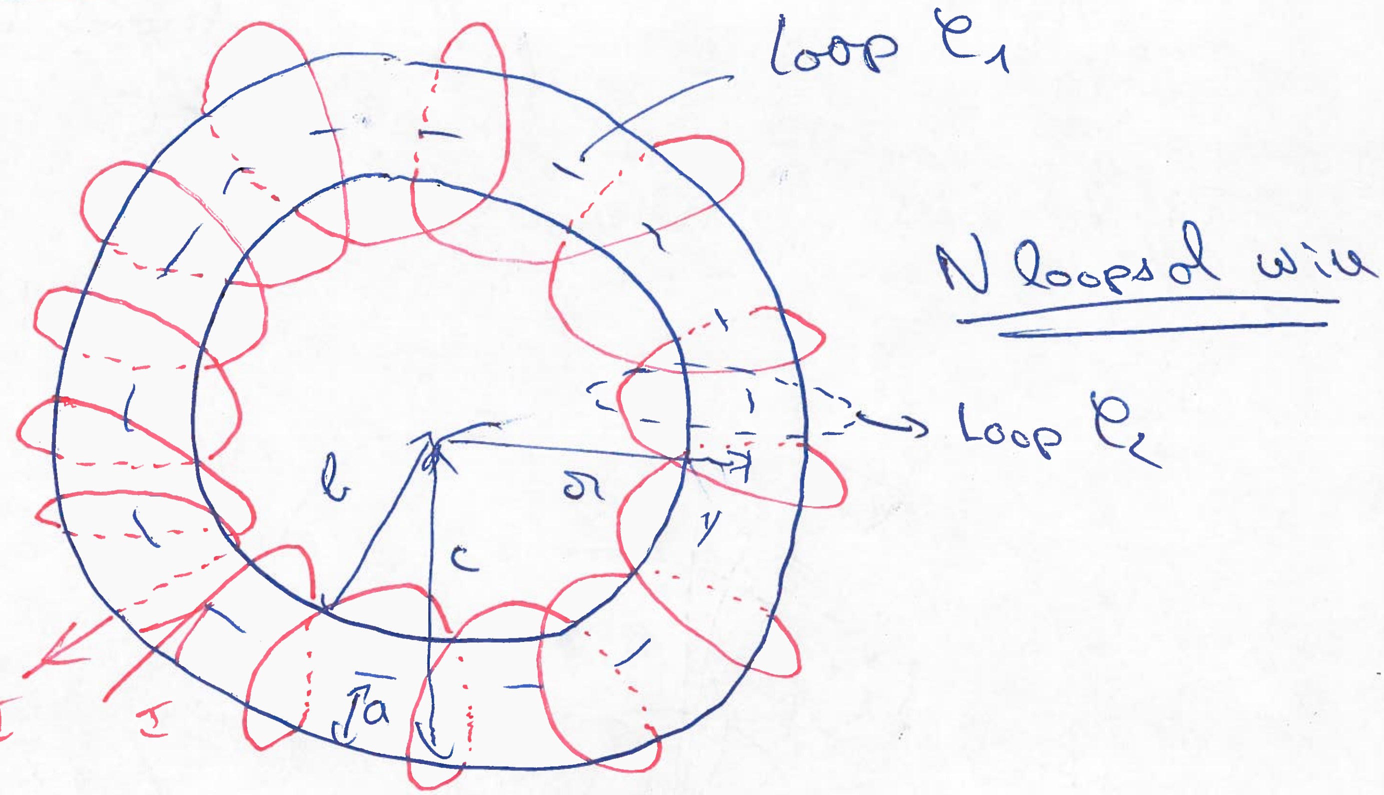
$$\frac{I'}{I} = \frac{\pi r^2}{\pi R^2} \Rightarrow I' = \frac{r^2}{R^2} I$$

$$\oint_{\mathcal{C}_2} \vec{B} \cdot \vec{dl} = B\,(2\pi r) = \mu_0 I' = \mu_0 \frac{r^2}{R^2} I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi R^2}\, r \qquad (r < R)$$



Magnetic field of a toroid



$$\oint_{C_1} \vec{B} \cdot d\vec{l} = B\,(2\pi r) = \mu_0 N I$$

$$\Rightarrow \vec{B} = \frac{\mu_0 N I}{2\pi r}$$

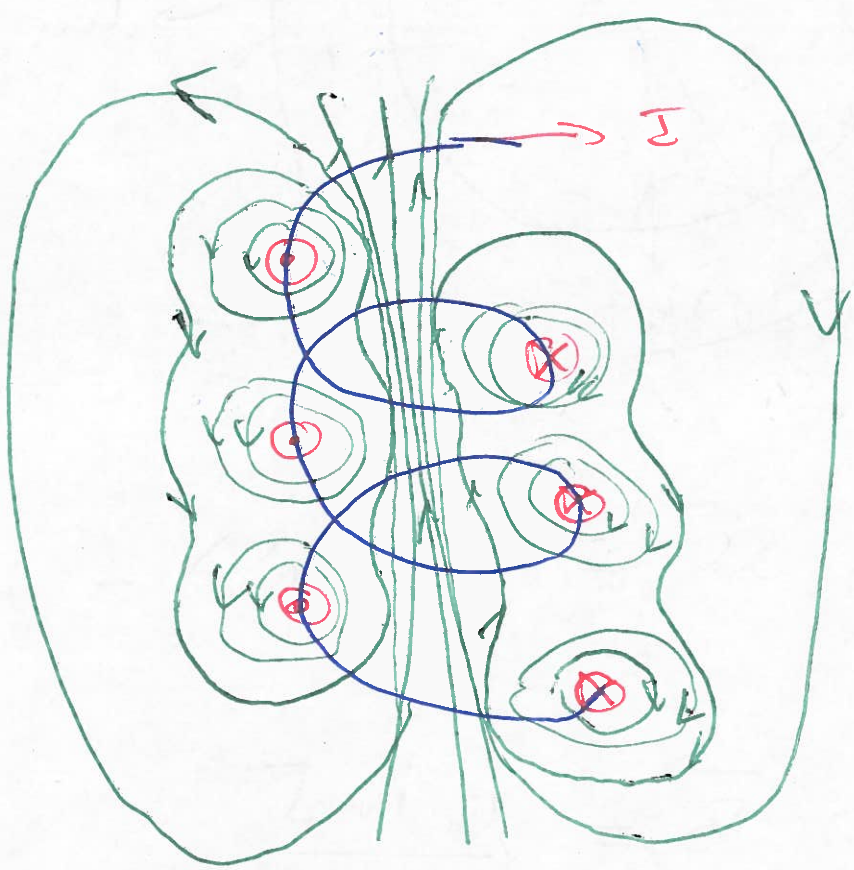
$B \sim$ uniform if $M \gg a$

Magnetic field outside

for $r < b \Rightarrow \oint_{C_1} \vec{B} \cdot \vec{dl} = 0$ but $\vec{B} \neq \vec{0}$

$\oint_{C_2} \vec{B} \cdot \vec{dl} = \mu_0 \underline{\underline{I}} \Rightarrow$ 1 wire passing through the loop C_2.

Magnetic field of a solenoid.



Bar magnet

+

uniform magnetic field inside

The ideal solenoid

$\vec{B}$ outside is weak (≈ 0)

$\vec{B}$ inside is uniform

$$\oint_{C_1} \vec{B}\cdot d\vec{l} = \mu_0 N I$$

on ② on ③ $\quad \vec{B} \perp d\vec{l} \Rightarrow \vec{B}\cdot d\vec{l} = 0$

on ④ $\Rightarrow B_{\text{outside}} \ll B_{\text{inside}} \Rightarrow$ neglect

$$\oint_{C_1} \vec{B}\cdot d\vec{l} = B \oint_{①} dl = B l$$

$$\Rightarrow \quad B = \mu_0 \frac{N}{l} I$$

n = # of loops per length

$$B = \mu_0 n I$$

Toroid when $r >> a \Rightarrow n = \frac{N}{2\pi r}$

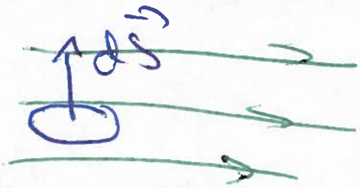
Ls of $B = \mu_0 \underbrace{\frac{N}{2\pi r}}_{= n} I = \mu_0 n I$

Gauss Law in magnetism

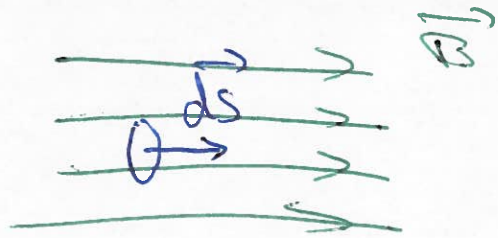
$$\phi_B = \int \vec{B} \cdot \vec{dS}$$

$$\vec{dS} = \vec{n}\, dS$$

$$d\phi_B = \vec{B} \cdot \vec{dS}$$
$$= B . dS \cos\theta$$

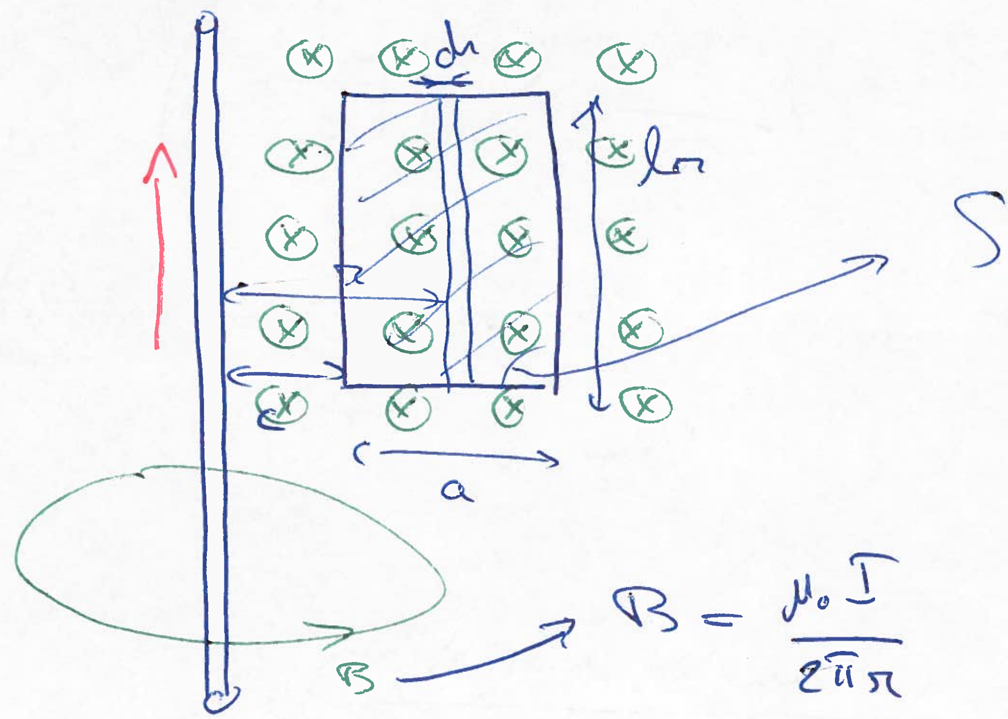


$$d\phi_B = \vec{B} \cdot \vec{dS} = 0$$



$$d\Phi_B = \vec{B} \cdot d\vec{S} = \underline{\underline{B\, dS}} \quad (\text{max flux})$$

Magnetic flux through a rectangular loop



$$B = \frac{\mu_0 I}{2\pi r}$$

$$\Phi_B = \int_S \vec{B} \cdot d\vec{S} = \int_S \vec{B} \cdot dS\, \vec{n}$$

$$= \int_S B\, dS = \int \frac{\mu_0 I}{2\pi r}\, dS$$

$$dS = b\,dr$$

$$\Phi_B = \int \frac{\mu_0 I}{2\pi r}\, b\, dr$$

$$= \frac{\mu_0 I b}{2\pi} \int_c^{a+c} \frac{dr}{r}$$

$$= \frac{\mu_0 I b}{2\pi} \ln\left(\frac{a+c}{c}\right)$$

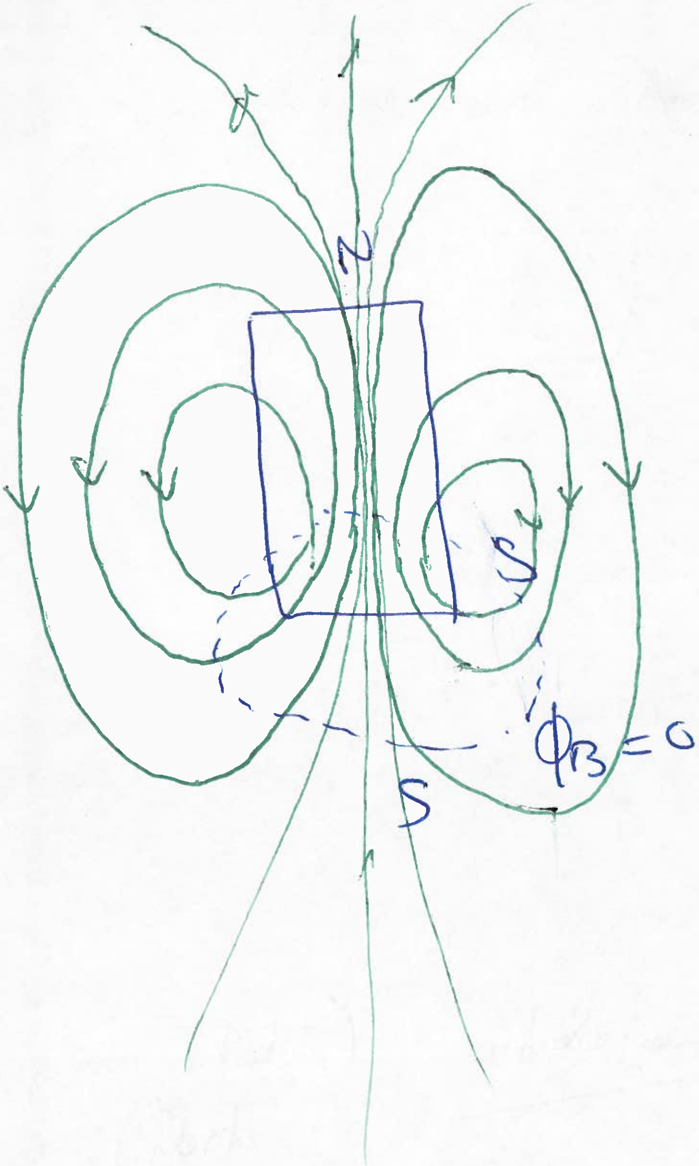
Any magnetic field line form a closed loop

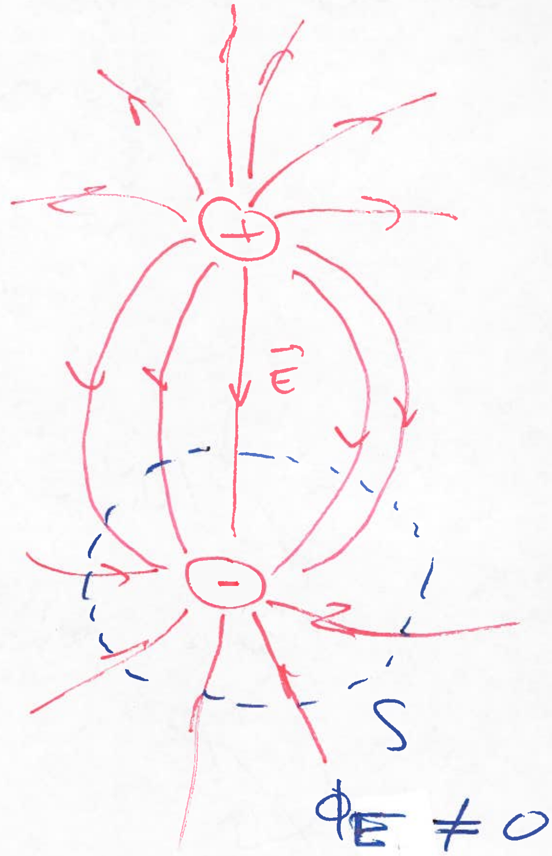
Gauss's law in magnetism

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

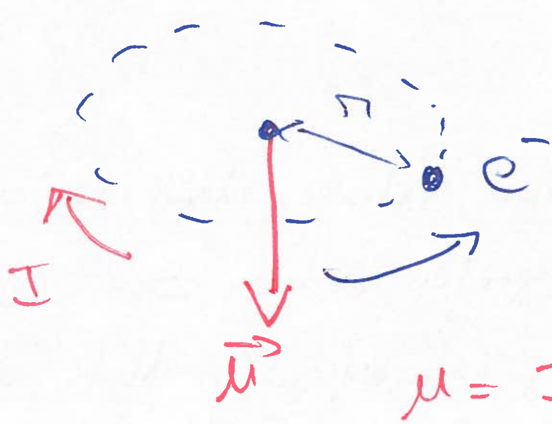
(closed surface)

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0}$$





Magnetism in matter



$\mu = IS \Rightarrow$ Magnetic moment

$= I \pi r^2$

Smallest non zero magnetic moment of an electron

$$\mu = \sqrt{2}\,\frac{e}{2m_e}\,\hbar$$

$\hbar = h/2\pi = 1.05 \cdot 10^{-34}$ J.s

Planck constant

Spin of an electron (rotation along itself)

$$\mu_{spin} = \frac{2\hbar}{2m_e}$$

Ferromagnetism : interaction between atoms ⇒ alignment of magnetic moments ⇒ creation of a strong magnetization that remains when the external field is removed substance in which

Paramagnetism: Magnetic moment is weak and in the same direction as the applied $\vec{B}$

Diamagnetism substance in which magnetic moment is weak and in opposite direction to the applied field $\vec{B}$