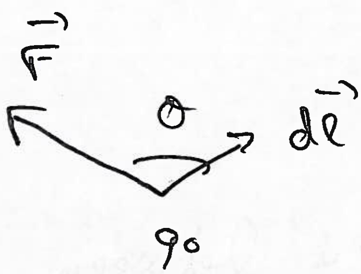
The electric potential

(Chapter 25)

Let's place a test charge q_0 in an electric field $\vec{E}$ $\Rightarrow$ force $\vec{F} = q_0 \vec{E}$

A displacement $\vec{dl}$ of this charge will produce a work : $dW = \vec{F} \cdot \vec{dl}$ $\Rightarrow$ 1 Joule

$1\,J = 1\,N.m$



$$dW = F\, dl \cos\theta$$

$dW < 0$: $\theta > \pi/2$ $\Rightarrow$ the operator moving q_0 is fighting against the force

$\Rightarrow$ energy won by the charge

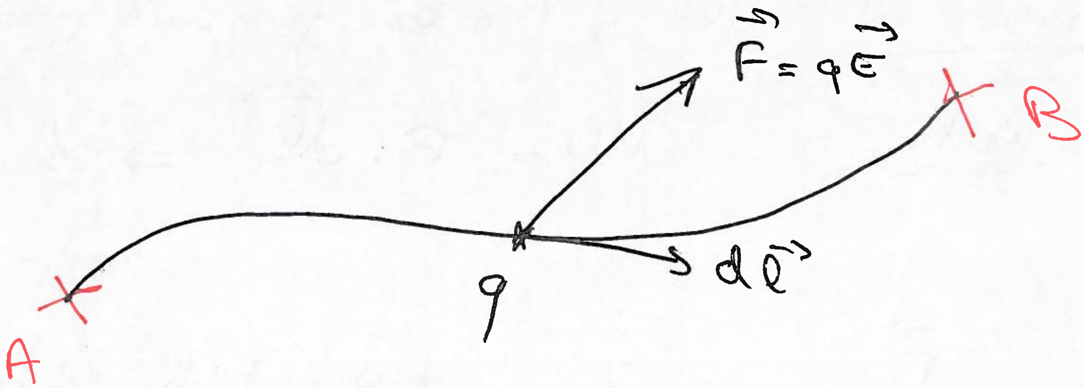
$dW > 0$ $\theta < \pi/2$ $\Rightarrow$ the force and the operator are acting in the same direction $\Rightarrow$ energy lost by the charge (↓ of potential energy)

Def Potential Energy $\boxed{dU_p = -dW}$

1 Work, force, energy, potential

Variation of electrostatic potential energy

$$dU_p = -dW = -q\,\vec{E}\cdot\vec{dl}$$



Variation of potential energy between two points A and B, following any path

$$U_p(B) - U_p(A) = \int_A^B dU_p = -\int_A^B q\vec{E}\cdot\vec{dl}$$

Definition : Electrical potential V = rate of potential energy per charge unit

$$V(P) = U_p(P)/q \Rightarrow U_p(P) = q\,V(P)$$

$\vec{F} = q\vec{E}$: $\vec{E}$ = rate of electrostatic force / charge unit

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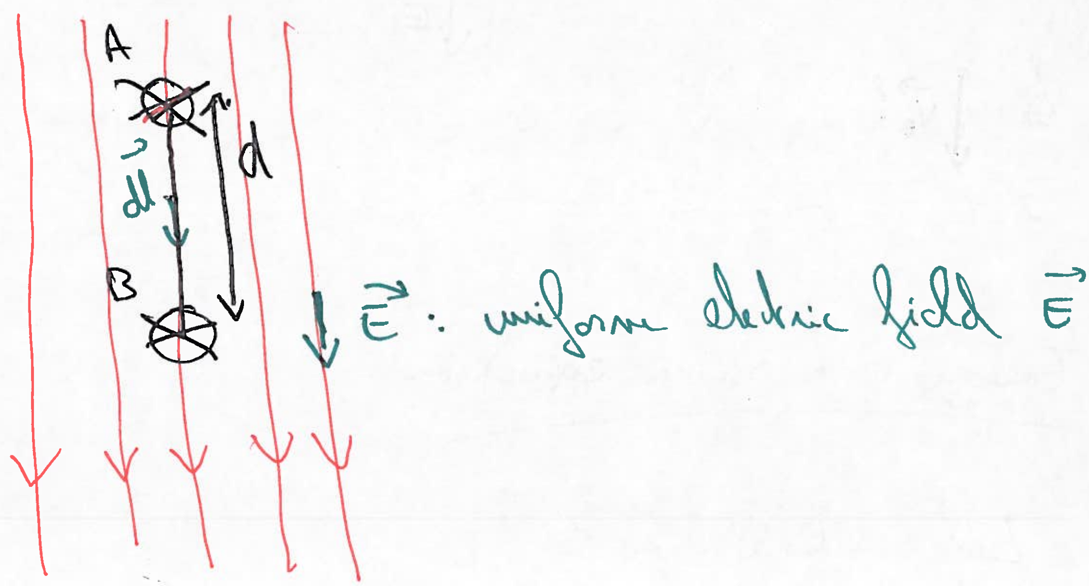
$U_p(P) = q\,V(P)$ V = rate of potential energy / charge unit

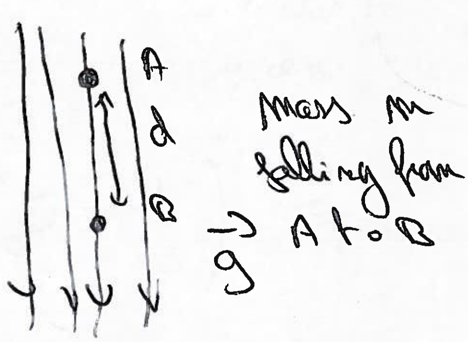
An electrical potential V exists even if the test charge q does not exist.

Motion of a charge in a uniform electric field $\vec{E}$



Positive charge moving from A to B

Analogy :



$$V_B - V_A = -\int_A^B \vec{E}\cdot\vec{dl}$$

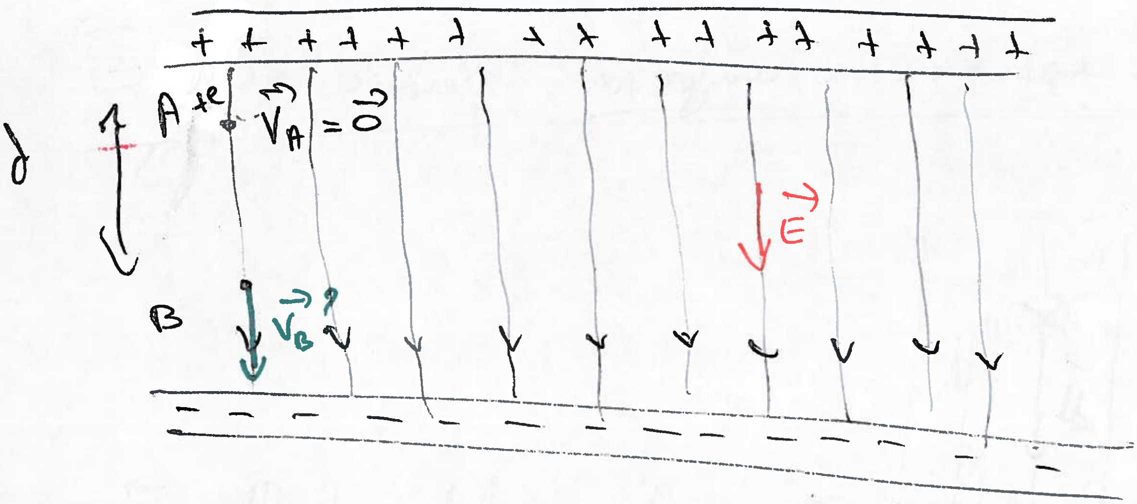
$$= -\int_A^B E\,dl\,\underbrace{\cos(0^\circ)}_{=1} = -Ed \qquad \Delta V = -Ed$$

Change in potential energy : $\Delta U_p = -qEd$

if $q > 0 \Rightarrow \Delta U_p < 0$ (operator an force in the same direction)

if $q < 0 \Rightarrow \Delta U_p > 0$ (operator fights against the force)

Motion of a proton in a uniform electric field $\vec{E}$



Energy conservation equation

$$\Delta K + \Delta U_p = 0$$

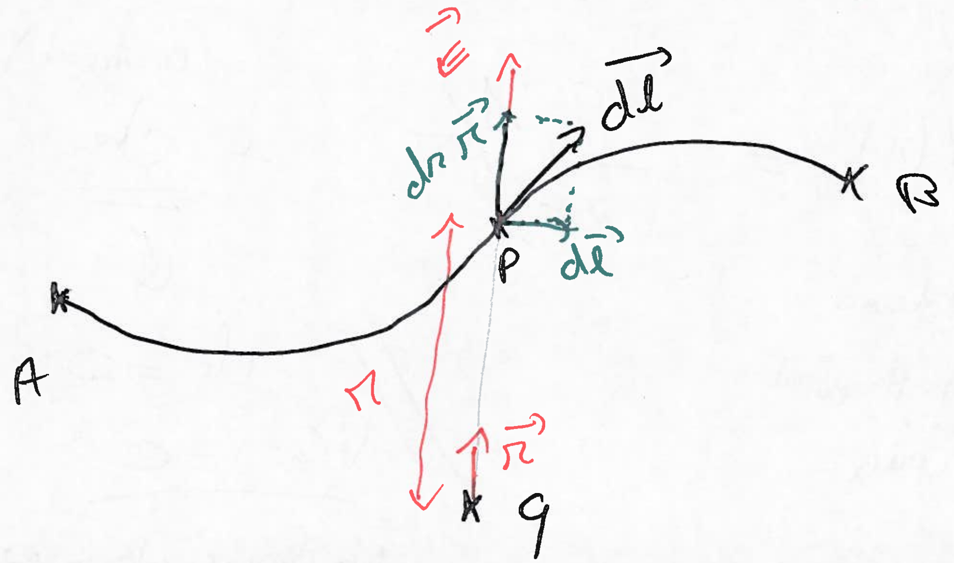
ΔK → Kinetic energy

$$\Rightarrow \left(\frac{1}{2} m v_B^2 - 0\right) + e \underbrace{\Delta V} = 0$$

$$\Delta V = -Ed$$

$$v_B = \sqrt{\frac{2eEd}{m}}$$

Potential created by a punctual charge



$$\vec{dl} = \vec{dl}_\perp + dr\,\vec{r}$$

$$\vec{E}(P) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^2}$$

$$\Rightarrow V(B) - V(A) = -\int_A^B \vec{E}\cdot\vec{dl}$$

$$= -\int_A^B \Big(\underbrace{\cancel{\vec{E}\cdot\vec{dl}_\perp}}_{=0} + \underbrace{\vec{E}\cdot dr\,\vec{r}}_{E\,dr}\Big)$$

$$= -\frac{q}{4\pi\epsilon_0}\int_{r_A}^{r_B}\frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_B}-\frac{1}{r_A}\right)$$

$$\frac{d(1/r)}{dr} = -1/r^2$$

if we assume no charge at $+\infty$

$$V(P) = V(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + \underset{\Downarrow}{\underline{\underline{cte}}}$$

(constant)

r: distance from the point charge

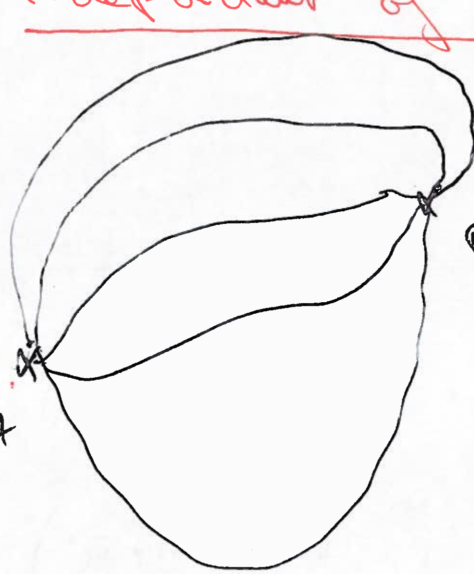
$cte = 0$

$\underline{V(\infty) = 0}$

No charge at $+\infty$ therefore $V(+\infty) = 0$ therefore the constante $= 0$

$$V(B) - V(A) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$\Rightarrow$ <u>independant of the path from A to B</u>



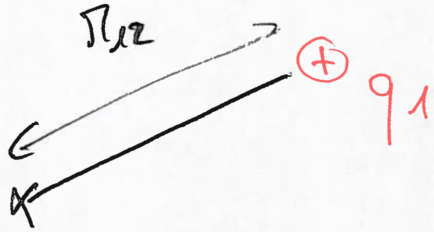
$$V(B) - V(A) = -\int_A^B \vec{E}\cdot\vec{d\ell}$$

$\Rightarrow$ <u>$\vec{E}$ is conservative</u>

$\Rightarrow$ <u>$\vec{F} = q\vec{E}$ is conservative</u>

$$\oint_A^A \vec{E}\cdot\vec{d\ell} = V(A) - V(A) = 0$$

Potential energy of a system of discrete charges



$$V_1(P) = \frac{k_e\, q_1}{r_{12}}$$: a charge q_1 creates an electric potential in P

Let's place a charge q_2 from $+\infty$ to the point P

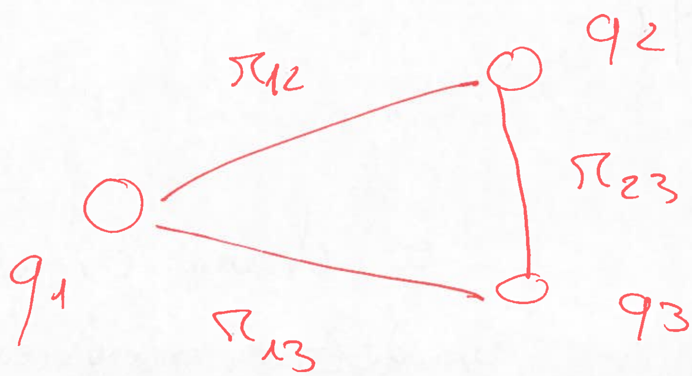
$\Rightarrow$ Variation of potential energy

$$\Delta U_P = \underbrace{U_P - 0}_{\text{0 potential energy when } q_2 \text{ is at } +\infty} = q_2 \, \Delta V$$

$$\Delta U_P = q_2 \, \Delta V = q_2 \left(k_e \frac{q_1}{r_{12}} - 0 \right)$$

$$\boxed{U_P = k_e \frac{q_1 q_2}{r_{12}}}$$

Potential energy of a system of 3 charges



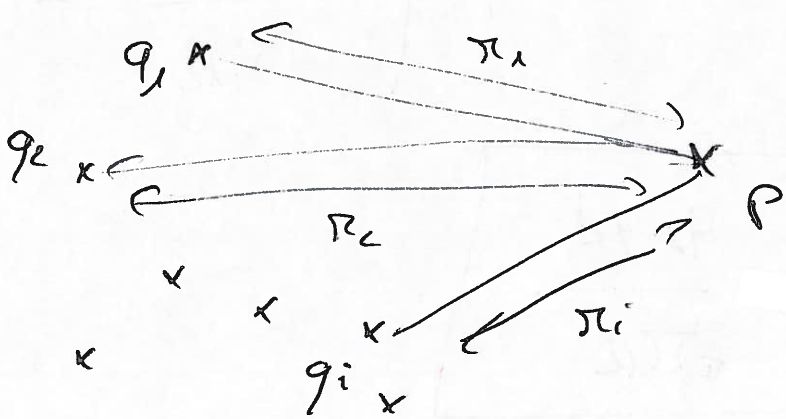
$$U_p = k_e \underbrace{\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)}_{\sum \text{ over all possible pairs.}}$$

$$\Rightarrow \boxed{U_p = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} k_e \frac{q_i q_j}{r_{ij}}} = \frac{1}{2} \sum_i q_i V_i$$

$V_i \rightarrow$ potential created at q_i by all the other charges q_j ($j \neq i$)

Estimation of the electrical potential

* From a distribution of punctual charges



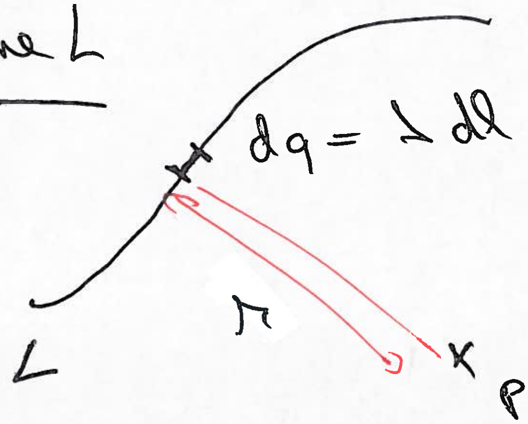
Superposition principle

$$\boxed{V(P) = \sum_i k_e \frac{q_i}{r_i}}$$

(for a finite distribution $\Rightarrow V(+\infty) = 0$)

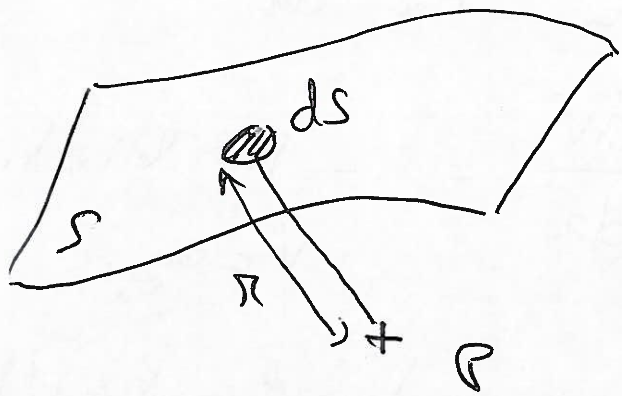
* Electrical potential created by a continuous distribution of charges

* along a line L



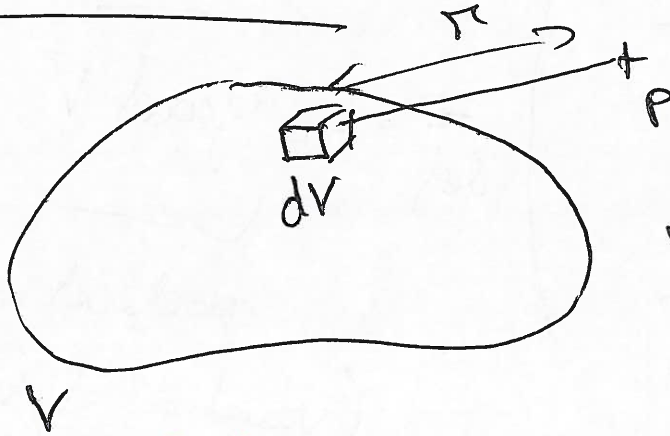
$$V(P) = \int_L k_e \frac{\lambda \, dl}{r}$$

* along a surface S



$$V(P) = \int_S k_e \frac{\sigma \, ds}{r}$$

* within a volume V

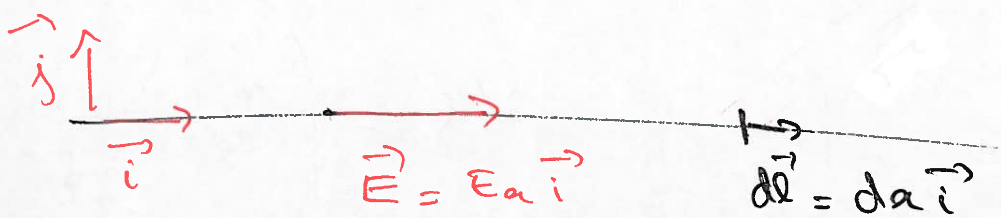


$$V(P) = \int_V k_e \frac{\rho \, dV}{r}$$

Scalar integral for V $\Rightarrow$ easier than for the vector $\vec{E}$

* How to compute $\vec{E}$ from the electrical potential V

let's assume $\vec{E} = E_x \, \vec{i}$ (along the x axis only)



$$dV = -\vec{E} \cdot \vec{dl} = -E_x \, dx$$

$$\Rightarrow E_x = -\frac{dV}{dx} = -\text{ the derivative of } V \text{ along } x$$

Generalization in 3D

$$\vec{E} = \begin{pmatrix} \underbrace{-\frac{\partial V}{\partial x}}_{\text{partial derivative along } x} \\ -\frac{\partial V}{\partial y} \\ -\frac{\partial V}{\partial z} \end{pmatrix} \underset{\text{def}}{\equiv} -\underbrace{\overrightarrow{\text{grad}}\, V}_{\text{gradient of } V}$$

gradient of V (variation of V in x, y and z directions)

Definition of an equipotential line

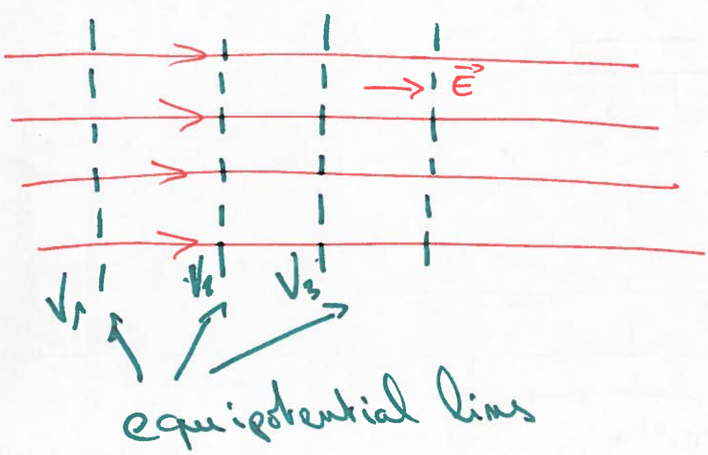
Equipotential line $\Rightarrow$ V is constant along the line

$\Rightarrow dV = 0$ along the line

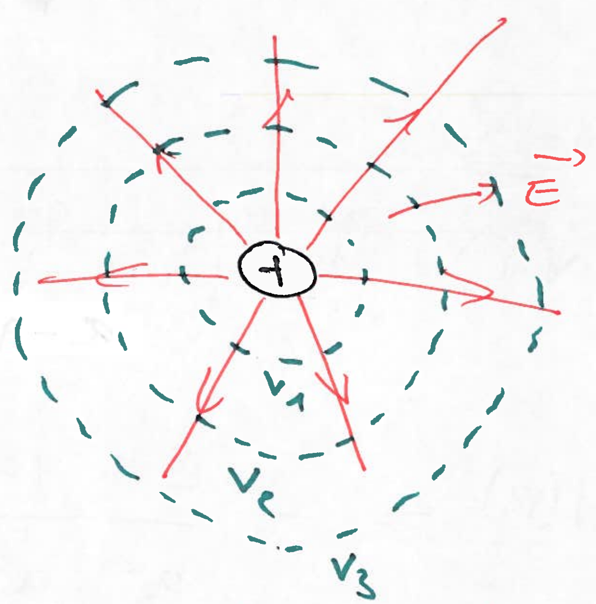
$\Rightarrow -\vec{E}\cdot\vec{dl} = 0$

$\Rightarrow \vec{E} \perp \vec{dl}$ $\Rightarrow$ equipotential lines are $\perp$ to line fields.

Uniform field

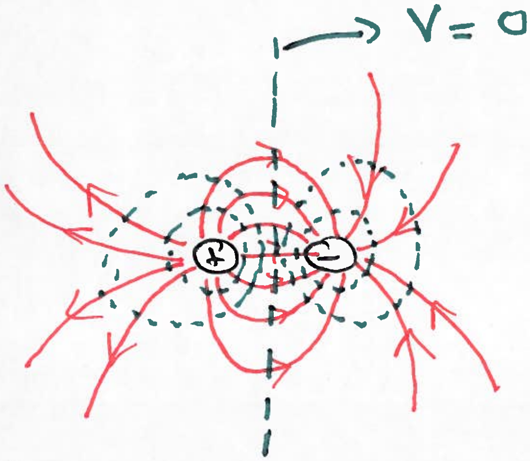


Field created by a punctual charge



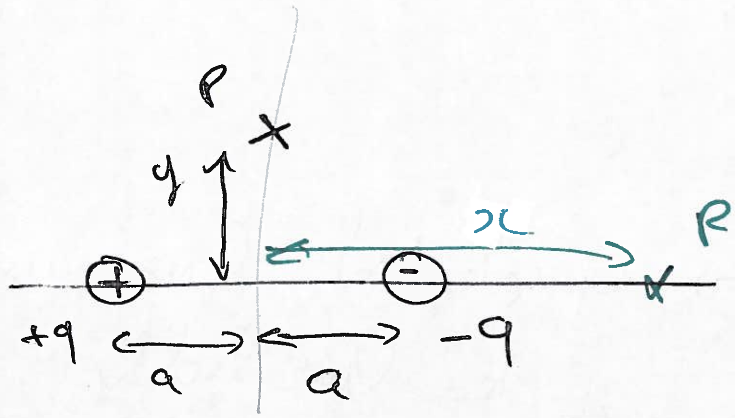
equipotential sphere

Electric dipole



Examples

* Electrical potential of an electric dipole



in P on the midline

$$V(P) = k_e \left(\frac{q}{\sqrt{a^2+y^2}} + \frac{-q}{\sqrt{a^2+y^2}} \right) = 0$$

in R

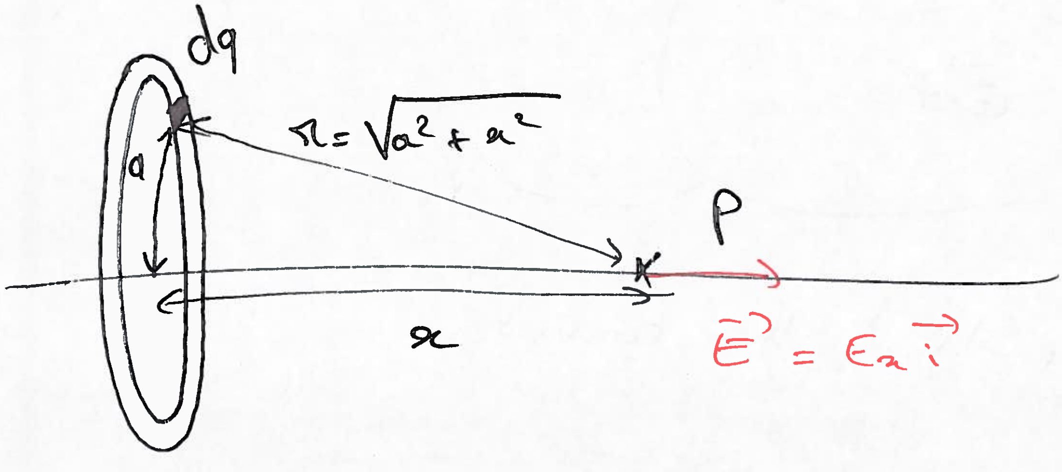
$$V(R) = k_e \left(\frac{-q}{x-a} + \frac{q}{x+a} \right)$$

$$V(R) = -\frac{2k_e q a}{x^2 - a^2}$$

if $x >> a$ $\boxed{V(R) \simeq -\frac{2k_e q a}{x^2}}$

$\vec{E} = E_x \vec{i}$ $\quad E_x = -\frac{dV}{dx} = -\frac{4k_e q a}{x^3}$

Electrical potential created by a ring uniformly charged



total charge of the ring

$$V(P) = k_e \int_{ring} \frac{dq}{\underset{\displaystyle = \sqrt{a^2+x^2}}{r}} = \frac{k_e}{\sqrt{a^2+x^2}} \int dq = \frac{k_e Q}{\sqrt{a^2+x^2}}$$

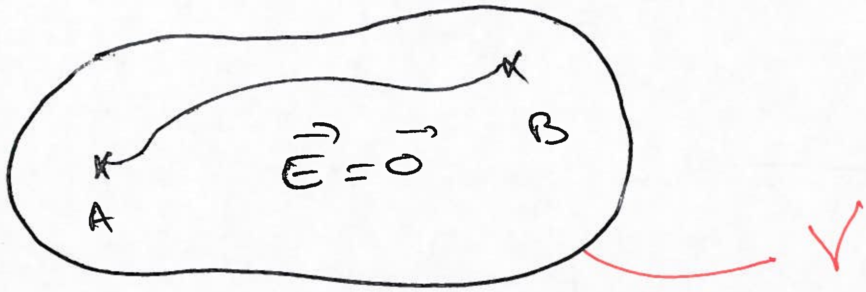
$\vec{E} = E_x \vec{i}$ with

$$E_x = -\frac{dV}{dx} = -k_e Q \frac{d}{dx} (a^2+x^2)^{-1/2}$$

$$E_x = -k_e Q \, (-1/2) \, (a^2+x^2)^{-3/2} \, (2x)$$

$$\vec{E} = \frac{k_e x}{(a^2+x^2)^{3/2}} Q \, \vec{i}$$

Electric potential of a conductor in equilibrium



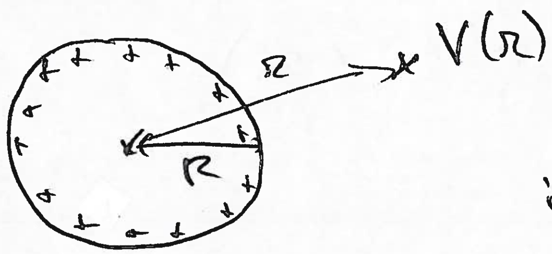
∀ A, B points of the conductor

$$V(B) - V(A) = -\underbrace{\int_A^B}_{\text{path inside the conductor}} \vec{E}\cdot d\vec{l} = 0$$

because $\vec{E} = \vec{0}$

⇒ <u>equipotential</u> : V = constante inside the conductor

Charged sphere (<u>conductor</u>)

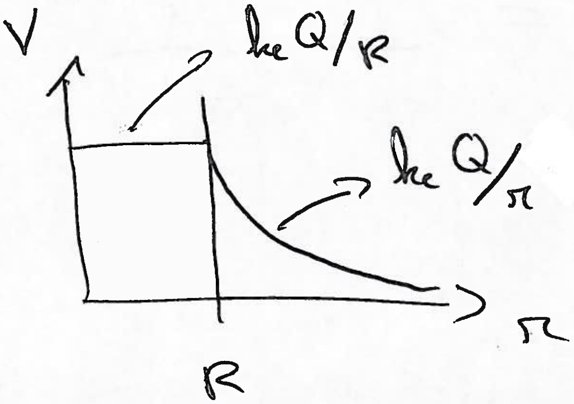


if $r > R$: the sphere of charge Q behaves like a punctual charge of charge Q

$$V(r) = \frac{kQ}{r} \quad \text{if } r > R$$

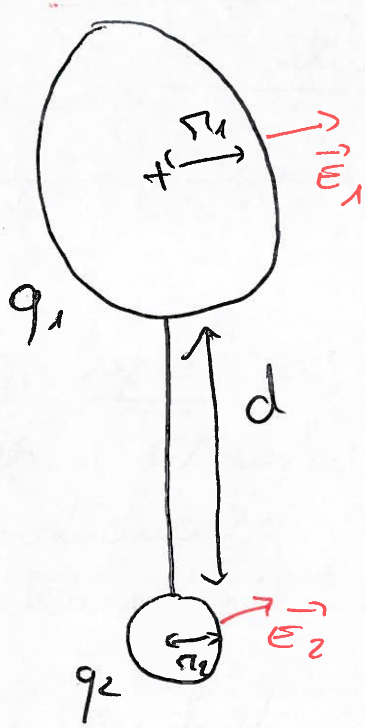
if $r < R$

$V(r)$ = constante inside the conductor $= k_e \dfrac{Q}{R}$ (continuity)



Two spheres connecting throug an electrical wire

$\Rightarrow$ same potential at equilibrium



if the two spheres are quite distant from each other

$$\left.\begin{array}{l} d \gg r_1 \\ d \gg r_2 \end{array}\right\} \quad \underbrace{V = k_e \frac{q_1}{r_1} = k_e \frac{q_2}{r_2}}$$

$$\Rightarrow \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

Electric field at the surface of the sphere (radial) $E_1 = k_e \dfrac{q_1}{r_1^2}$ $\quad E_2 = k_e \dfrac{q_2}{r_2^2}$

Therefore

$$\frac{E_1}{E_2} = \underbrace{\frac{q_1}{q_2}}_{\substack{\shortparallel \\ r_1/r_2}} \frac{r_2^{\,2}}{r_1^{\,2}} = \frac{r_1}{r_2}\,\frac{r_2^{\,2}}{r_1^{\,2}}$$

because the 2 spheres are connected

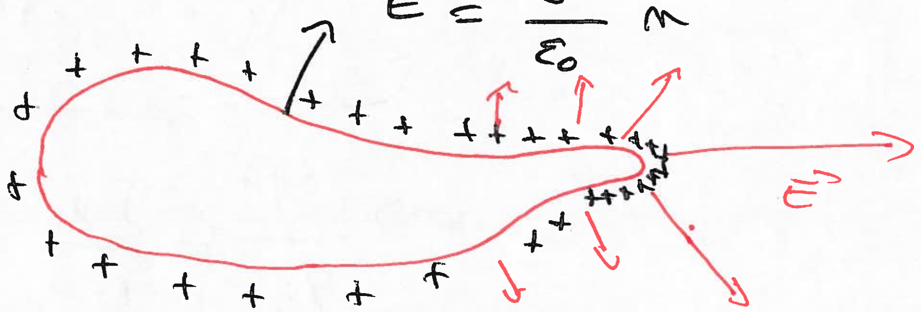
$$\Longrightarrow \frac{E_1}{E_2} = \frac{r_2}{r_1}$$

so if $r_2 \to 0 \Longrightarrow E_2 \to +\infty$

$\Longrightarrow$ the field is large where the radius of curvature is smaller

Generalization

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{n}$$

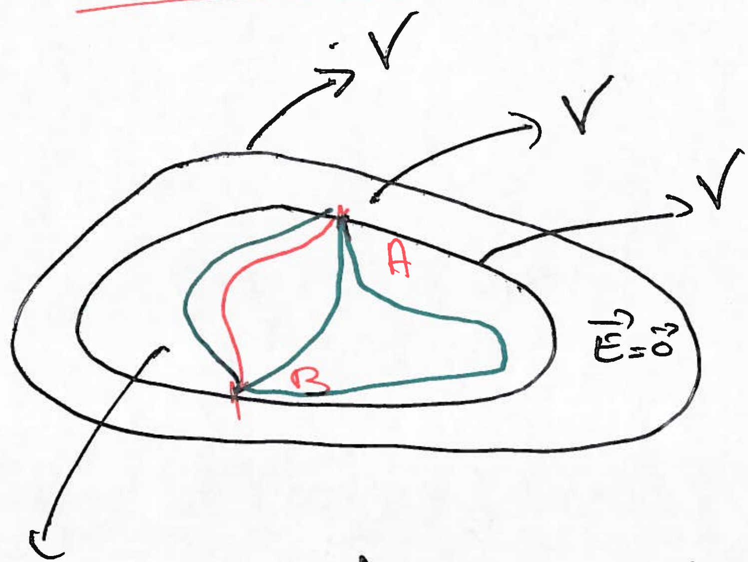


σ is larger where the radius of curvature is smaller

$\Longrightarrow$ Corona discharge

$\Longrightarrow$ Electrostatic wind

Electric potential within the cavity of a conductor free of charges



Conductor in equilibrium

$\Downarrow$

V = constant

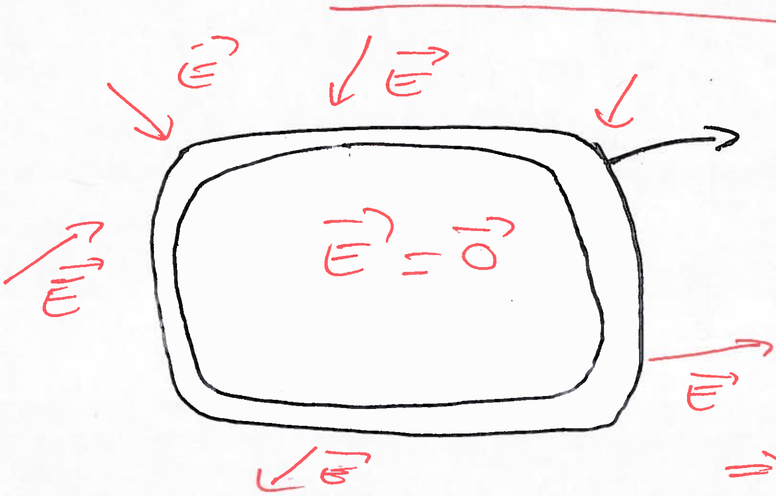
cavity $\Rightarrow$ no charge in cavity

$$V(B) - V(A) = 0 = -\int_A^B \vec{E} \cdot \vec{dl}$$

for any path between A and B

$$V(B) = V(A) = V$$

$\Rightarrow$ $\vec{E} = \vec{0}$ inside the cavity



conductor in equilibrium at potential V

$\vec{E}$ inside $= \vec{0}$ whatever is the field outside

$\Rightarrow$ Faraday cage.