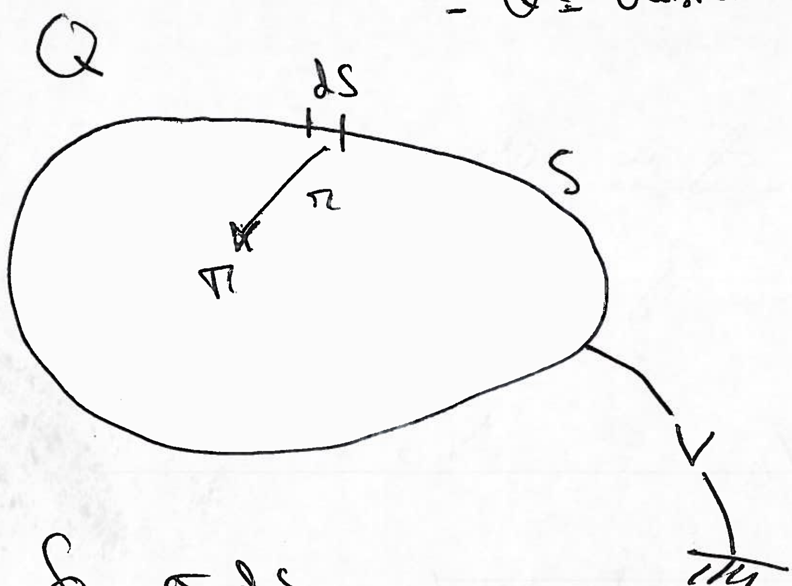
Capacitance and dielectrics

(Chap 26)

Conductor with charge Q and potential V

At equilibrium :
- $\vec{E} = \vec{0}$ inside the conductor
- V = constant
- Q = distributed on the surface ($\sigma\, dS$)



$$Q = \oint_S \sigma\, dS$$

$$V = \oint_S \frac{1}{4\pi\varepsilon_0} \frac{\sigma\, dS}{r}$$

if $V \to \lambda V$ (we apply another electrical potential on the conductor)

$\Rightarrow \sigma \to \lambda \sigma \Rightarrow Q \to \lambda Q$

$\Rightarrow Q/V = \dfrac{\lambda Q}{\lambda V} = \underline{\text{constant}}$

Capacitance $\boxed{C = \dfrac{Q}{\Delta V}}$

$C > 0$ by convention

Unit SI : 1 Farad $= 1\,F = 1\,C/V$

Typical capacitance $1\,\mu F \rightarrow 1\,pF$

$10^{-6}F \qquad 10^{-12}F$

Capacitance of a sphere : conductor of charge Q

↳ sphere of radius a

$V = k_e Q/a \qquad V_\infty = 0$

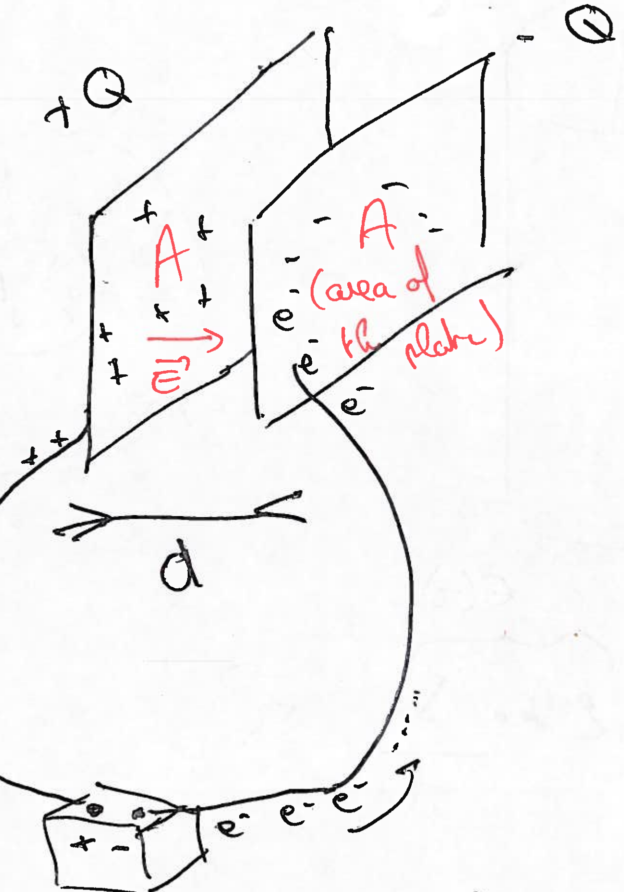
$\Delta V = V - V_\infty = k_e Q/a$

$$C = \frac{Q}{\Delta V} = \frac{Q}{k_e Q/a} = \frac{a}{k_e} =$$

$$\boxed{C = 4\pi \varepsilon_0 a}$$

We assume ε_0 to be the same in free space and in a conductor

Parallel plate capacitors



Battery : source of ΔV

Plate very close together : uniform field

$$\vec{E} = \frac{\sigma}{\varepsilon_0} \vec{i}$$

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \qquad (\sigma = Q/A)$$

$$\Delta V = Ed = \frac{Qd}{\varepsilon_0 A}$$

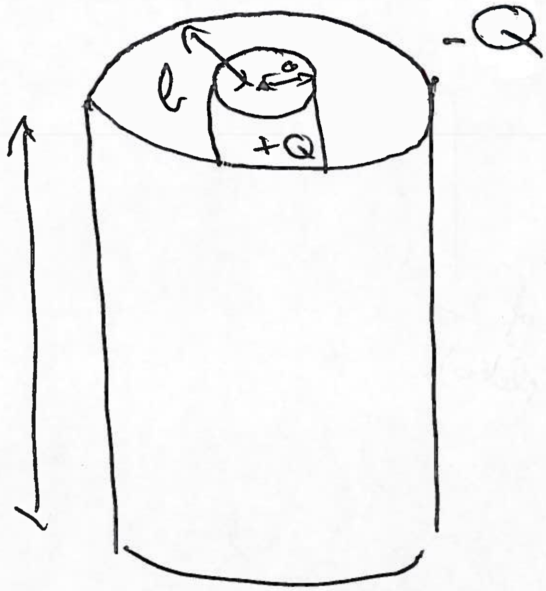
(E is uniform)

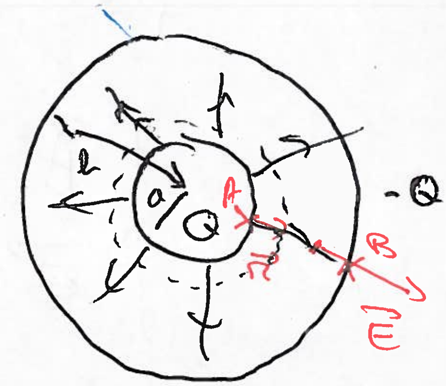
$$\Rightarrow \boxed{C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d}}$$

Cylindrical capacitor

Lineic charge density

$$\lambda = Q/l$$





$$\vec{E} = \underbrace{\frac{2k_e\lambda}{r}}^{E(r)} \vec{r}$$

$$\Delta V = V_B - V_A < 0$$

$$= -\int_A^B \vec{E}\cdot d\vec{l}$$

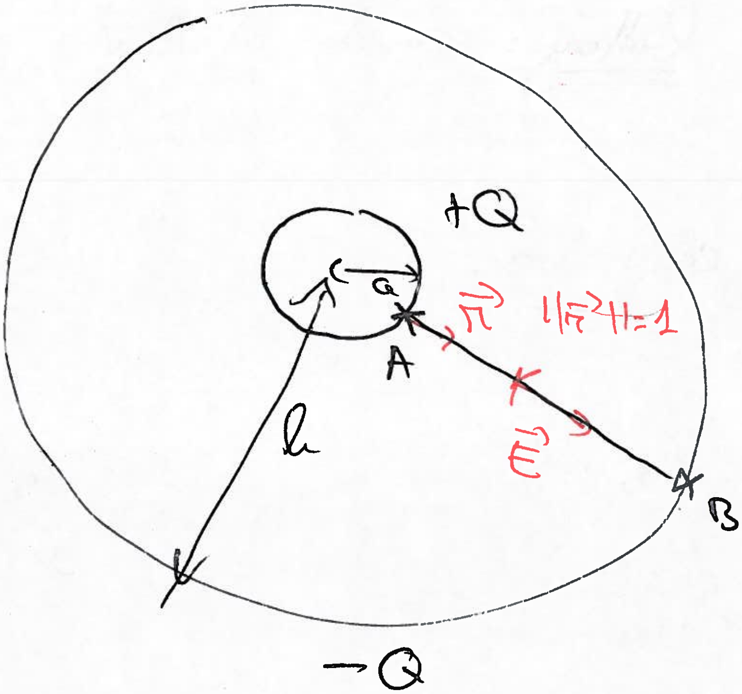
$$= -\int_A^B E(r)\,dr \quad \Rightarrow \quad C = \frac{Q}{|\Delta V|}$$

$$= -2k_e\lambda \int_a^b \frac{dr}{r} \qquad C = \frac{Q}{(2k_e Q/l)\ln(b/a)}$$

$$= -2k_e\lambda \ln(b/a) \qquad \boxed{C = \frac{l}{2k_e \ln(b/a)}}$$

$\Rightarrow$ coaxial cable

Spherical capacitor



$$\vec{E} = \frac{k_e Q}{r^2} \vec{r} \quad \text{for} \quad a < r < b$$

(Gauss law)

$$V_B - V_A = -\int_A^B \vec{E} \cdot \vec{dl}$$

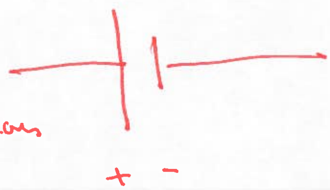
$$= -\int_A^B E(r)\, dr = -k_e Q \int_A^B \frac{dr}{r^2}$$

$$= k_e Q \left(\frac{1}{b} - \frac{1}{a} \right) = k_e Q \frac{(a-b)}{ab} < 0$$

$$C = \frac{Q}{|\Delta V|} = \frac{ab}{k_e (b-a)}$$

Combination of capacitors

Symbolic representation



Battery : generator of ΔV

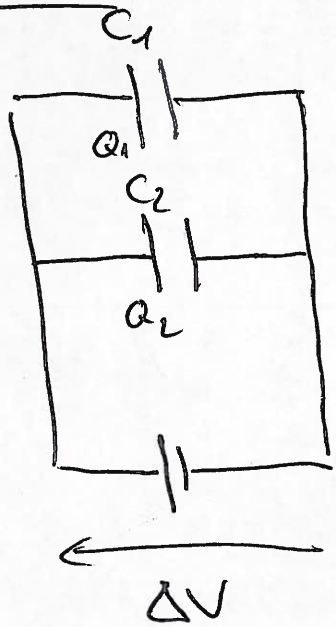
capacitor

close

open

switch

Capacitors in //



$$\Delta V = \Delta V_1 = \Delta V_2$$

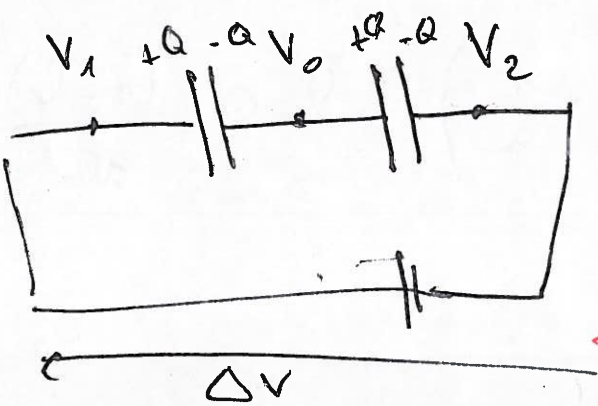
$$Q_{tot} = Q_1 + Q_2$$

$$C_{eq} \Delta V = C_1 \Delta V_1 + C_2 \Delta V_2$$

$$\boxed{C_{eq} = C_1 + C_2 + \dots}$$

Capacitors in series

$$Q = Q_1 = Q_2$$



$$V_1 - V_0 = Q/C_1$$

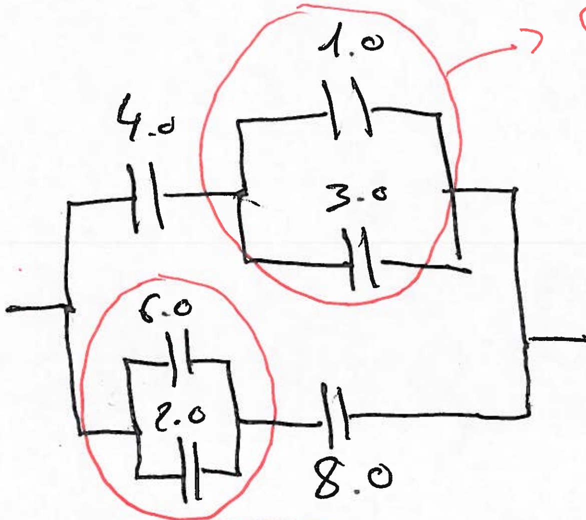
$$V_0 - V_2 = Q/C_2$$

$$V_1 \cancel{- V_0 + V_0} - V_2 = Q/C_1 + Q/C_2$$

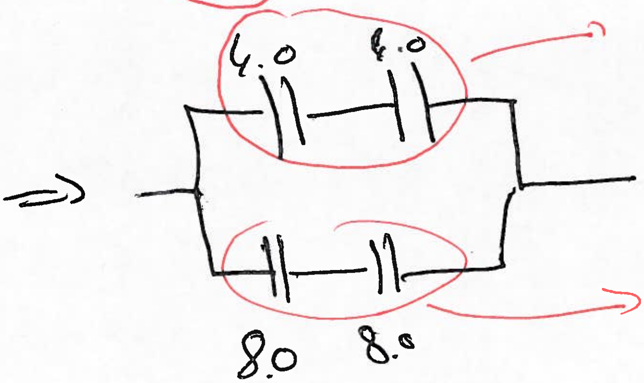
$$\boxed{1/C_{eq} = 1/C_1 + 1/C_2 + \dots}$$

Equivalent capacitance

eg:

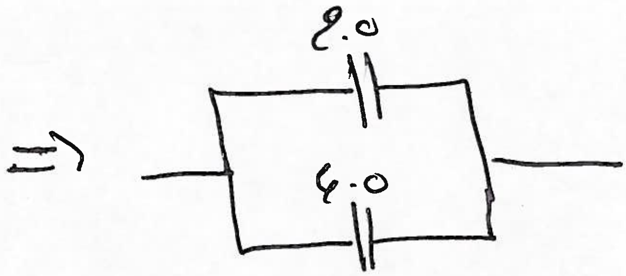


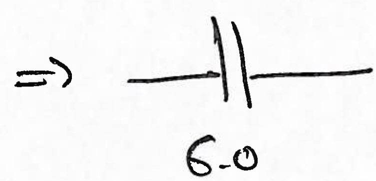
$C = 1 + 3 = 4.0$



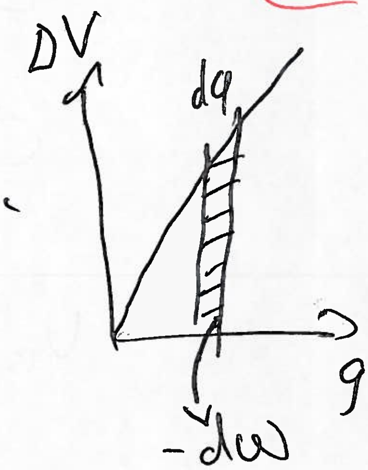
$1/C_{eq} = 1/4 + 1/4 = 1/2$

$1/C_{eq} = 1/8 + 1/8 = 1/4$





Energy stored in a charge capacitor



Let's bring manually dq to one plate.

$$dU_p = -dW = \Delta V \, dq = \frac{q}{C} \, dq$$

$$U_p = \int_0^Q \frac{q}{C} \, dq = \frac{1}{C} \int_0^Q q \, dq = \frac{1}{2} \frac{Q^2}{C}$$

U_p

Potential energy stored by a capacitor :

$$U_p = \frac{Q^2}{2C} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2$$

Note

energy of punctual charges

$$\Rightarrow U_p = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} k_e \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i q_i \underbrace{V_i}_{\text{potential created at } q_i \text{ by all other charges}}$$

$$\Rightarrow U_p = \frac{1}{2} \int V(P) \overbrace{dq}^{\rho \, dv} = \frac{1}{2} V Q$$

the potential for a conductor in equilibrium

$$\Delta V = E d$$

$$C = \varepsilon_0 \frac{A}{d}$$

$$U_p = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)(E d)^2$$

$$U_p = \frac{1}{2}(\varepsilon_0 A d)\, E^2$$

Energy density : U_p / Ad : energy / unit of volume

$$u_p = \frac{1}{2}\varepsilon_0 E^2$$

Capacitors with dielectric

dielectric constant

$$\Delta V = \frac{\Delta V_0}{K} \qquad \Delta V < \Delta V_0 \qquad K > 1$$

$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{\Delta V_0 / K} = K \frac{Q_0}{\Delta V_0}$$

$$\boxed{C = K C_0 \qquad C > C_0}$$

example of dielectric

Paper : $K = 3.7$

Porcelain : $K = 6$

Water : $K = 80$

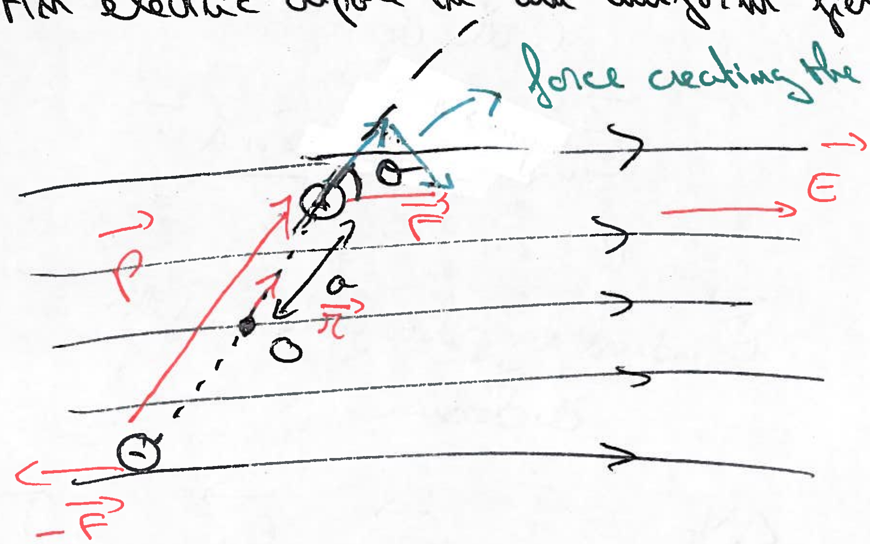
$$C = K \varepsilon_0 \frac{A}{d}$$

There is no net motion of charges in a dielectric but you have an influence of the electric field

$\Rightarrow$ polarization

↳ elements act as small electric dipoles

An electric dipole in an uniform field $\vec{E}$



force creating the rotation of the dipole $= F \sin\theta$

$\Downarrow$

torque $F \sin\theta \cdot a$

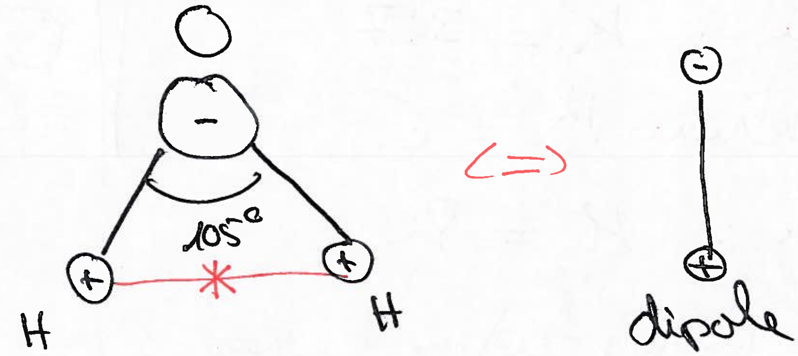
Moment of an electric dipole

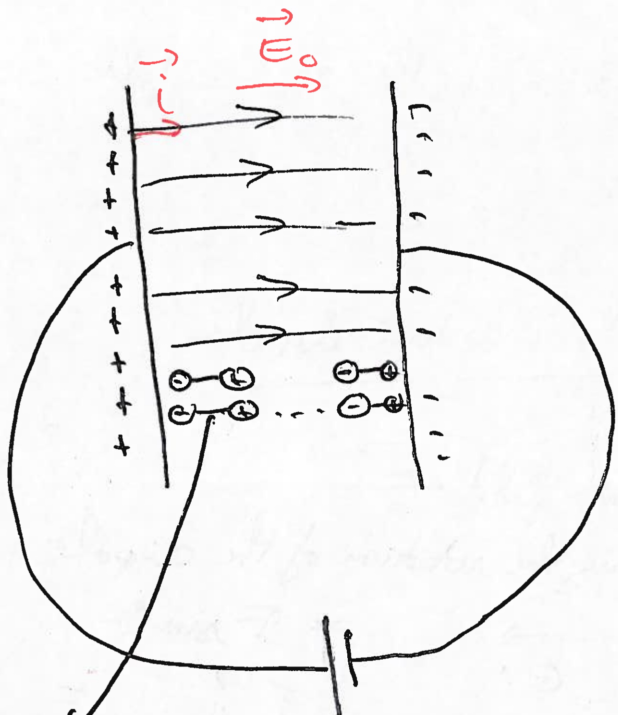
$\vec{p} \equiv 2aq\,\vec{n}$ $\Rightarrow$ Torque $\vec{\tau} = \vec{p} \times \vec{E}$

total torque $\tau = 2Fa \sin\theta$

with $F = qE \Rightarrow \tau = 2aq\,E\sin\theta = pE\sin\theta$

Example: water molecule





polarization of the molecule of the dielectric

$$\vec{E_0} = \frac{\sigma}{\varepsilon_0}\vec{i}$$

$\Rightarrow$ charge accumulation by the dipole of the dielectrique will create

$$\vec{E}_{ind} = -\frac{\sigma_{ind}}{\varepsilon_0}\vec{i}$$

$\vec{E}$ induced by polarization of the dielectric

Since $\Delta V = \dfrac{\Delta V_0}{K} \Rightarrow \vec{E} = \dfrac{\vec{E_0}}{K} = \dfrac{\sigma}{K\varepsilon_0}\vec{i}$

$$\vec{E} = \vec{E_0} + \vec{E}_{ind} = \left(\frac{\sigma}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0}\right)\vec{i}$$

$$\frac{\sigma}{K\varepsilon_0}\vec{i} \Rightarrow \frac{\sigma}{K\varepsilon_0} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0} \Rightarrow \boxed{\sigma_{ind} = \left(\frac{K-1}{K}\right)\sigma}$$