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1. Calculate using $z = 2 - i$ and $w = 1 + 2i$

[1] a) $z + \overline{w}$

Answer: $= (2 - i) + (1 - 2i) = 3 - 3i.$

[1] b) $z\,w$

Answer: $= (2 - i)(1 + 2i) = 2 + 4i - i - 2i^2 = 4 + 3i.$

[2] c) $\frac{z}{w}$

Answer: $= \frac{2-i}{1+2i}\frac{1-2i}{1-2i} = \frac{2-4i-i+2i^2}{1+4} = \frac{-5i}{5} = -i.$

[3] 2. Prove that $\overline{(zw)} = (\overline{z})\,(\overline{w})$ for any $z, w \in \mathbb{C}$. (note: an example is worth zero marks, you must prove this statement in general.)

Answer: First, we need our variable complex numbers: $z = a + bi$ and $w = c + di$. First: LHS

$$\begin{aligned}\overline{(zw)} &= \overline{(ac - bd) + (ad + bc)i} \\ &= (ac - bd) - (ad + bc)i.\end{aligned}$$

RHS:

$$\begin{aligned}(\overline{z})(\overline{w}) &= (a - bi)(c - di) \\ &= ac + bdi^2 - bci - adi \\ &= (ac - bd) - (ad + bc)i.\end{aligned}$$

LHS = RHS, so proven.

[2] 3. a) Find $b \in \mathbb{R}$ such that $\begin{bmatrix}1\\b\\2\end{bmatrix}$ is orthogonal to $\begin{bmatrix}-1\\1\\3\end{bmatrix}$.

Answer:
Orthogonal means $\begin{bmatrix}1\\b\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\3\end{bmatrix} = -1 + b + 6 = 0$ so $b + 5 = 0$ and $b = -5$.

[2] b) Calculate the projection of $\begin{bmatrix}-4\\2\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Answer:
By the definition,

$$= \frac{\begin{bmatrix}-4\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix}} \begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{-4 - 2 + 0}{1 + 1 + 0} \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}-3\\3\\0\end{bmatrix}.$$

[3] 4. Calculate the intersection (or confirm the lack of intersection) between the lines $\left\{ \begin{bmatrix} 0 \\ -4 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} a, \quad a \in \mathbb{R} \right\}$ and $\left\{ \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} b, \quad b \in \mathbb{R} \right\}$.

Answer:

Equating the two leads to the problem

$$\begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} b + \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} a$$

so $1 = -1a$, $-7 = 2b + a$ and $6 = -b - 4a$. The first leads to $a = -1$, so $-6 = 2b$ and $b = -3$. Now we need to double check the last one: we need $-b - 4a = 3 + 4 = 7$ to be equal to 6, and it is not. As a result, they do not intersect.

[4] 5. Is $\begin{bmatrix} 7 & 0 \\ 2 & 3 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \right\}$?

Answer:

$$\begin{bmatrix} 7 & 0 \\ 2 & 3 \end{bmatrix} = a \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

leads to

$$7 = -2a - 3b + 4c \qquad 0 = a - b \qquad 2 = a + b \qquad 3 = c.$$

So, $c = 3$. That's one value. Next, we get $a = b$ from the second, so $2 = 2a$ and $a = 1 = b$. Finally, we check the first equation, for

$$LHS = 7 \qquad RHS = -2 - 3 + 12 = 7$$

so it is in the span.

[4] 6. Using the definition, determine if the following set S is linearly independent or linearly dependent.

$$S = \left\{ 1 + t + t^3, \, 1 + t^2 + t^3, \, t + t^2 \right\}$$

Answer:

As usual:

$$a(x^3 + x + 1) + b(x^3 + x^2 + 1) + c(x^2 + x) = 0$$

so

$$(a + b)x^3 + (b + c)x^2 + (a + c)x + (a + b) = 0.$$

So, we get

$$a + b = 0 \qquad b + c = 0 \qquad a + c = 0 \qquad a + b = 0$$

so $a = -b$ and $-b + c = 0$ so $b = c$. If we substitute that into the second equation we get $2b = 0$ so $b = 0$. Next, we get $a = 0$ and $c = 0$ from that, so they are linearly independent.

[4] 7. Consider the collection of vectors in $\mathbb{R}^4$. $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid \begin{array}{c} a - 3c = 0 \\ b + c + d = 0 \end{array} \right\}$.

a) Show that U is a subspace of $\mathbb{R}^4$.

Answer:
This CAN be done the old fashioned way, with the subspace test, but that's the hard way. The general form of the vectors in the set:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3c \\ -c-d \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} d, \qquad c, d \in \mathbb{R}.$$

This means that

$$U = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and U must be a subspace since all spans are subspaces.

b) Find a basis for U. Be sure to justify that your set *is* a basis for U.

Answer:
Well, we have a spanning set, all we need to do is show it's linearly independent.

$$a \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so $a = 0$ and $b = 0$ using the last two rows. Done.

[6] 8. In each case, the given set is *not* a basis of the given vector space. Prove this for each one. You may use any of the methods that we saw in class, but be sure to justify!

a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

Answer:
The vector space is two dimensional, the set has three elements, they can not be linearly independent, and so can't be a basis.

b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} \right\}$ for $\mathcal{M}_{2\times 2}$.

Answer:
The vector space is 4 dimensional, the set has three elements and can't be a spanning set, and so not a basis.

c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$ for $\mathbb{R}^3$.

Answer:
This one has three elements in a three dimensional space, so we need to double check independence.

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $a + 2c = 0$, $a + 2c = 0$ and $b + 3c = 0$. This means that $a = -2c$, $a = -2c$ and $b = -3c$. This makes c a free variable, and so we have more than the trivial solution, the set is linearly dependent and done.

9. Consider the following candidate for a vector space: the set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1, x_2 \in \mathbb{R}$ and the operations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 2 \\ x_2 + y_2 + 4 \end{bmatrix}, \qquad a \boxtimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + 2a - 2 \\ ax_2 + 3a - 3 \end{bmatrix}.$$

[2] a) Confirm that the vector $\mathbf{z} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ operates as the zero vector for the $\boxplus$ operation.

Answer:
This one is best done with an arbitrary vector: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \boxplus \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x_1 - 2 + 2 \\ x_2 - 4 + 4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

making $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$ the zero vector in this space.

Using a specific vector (i.e., an example) rather than an arbitrary one is worth fewer marks. Not much fewer, though.

[2] b) Calculate $0 \boxtimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Answer:
Just calculate:

$$0 \boxtimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 + (2)(0) - 2 \\ 0x_2 + (3)(0) - 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

[2] c) What do the previous two results tell you about our candidate for a vector space?

First, if this is a vector space then $\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \mathbf{0}$ (the zero vector), since it fulfills the requirement. Also, if it is a vector space, then $0 \boxtimes \mathbf{x} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \mathbf{0}$. This is a contradiction, since the zero vector is unique. You can't have two of them in a vector space, so the given set, addition and multiplication, do NOT form a vector space.

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