

MATH3705 A — Test 4

Name and Student Number:

Total points: 20. No partial marks for Questions 1-3.

Closed book! Non-programmer calculators are allowed!

- [2] 1. Consider the wave equation $u_{tt} = c^2 u_{xx}$, $0 < x < L, t > 0$, subject to the boundary conditions $u(0, t) = 0, u(L, t) = 0$ and the initial conditions $u(x, 0) = f(x), u_t(x, 0) = g(x)$. The solution is given by:

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right),$$

where:

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Given the wave equation $u_{tt} = 5u_{xx}$, $0 < x < 2, t > 0$, subject to the boundary conditions $u(0, t) = u(2, t) = 0$ and the initial conditions $u(x, 0) = \sin(2\pi x), u_t(x, 0) = 3 \sin(4\pi x)$, find a_4 .

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution: b.

cf. Part IV, Exercise 13.

- [2] 2. Find a polynomial solution of the Laplace equation $u_{xx} + u_{yy} = 0$, $0 < x < 1, 0 < y < 1$ with boundary conditions $u(x, 0) = x, u(x, 1) = 0, u(0, y) = 0, u(1, y) = 1 - y$.

- (a) x (b) $x - y$ (c) $1 - y$ (d) $x + y - xy$ (e) $x - xy$

Solution: e.

Note that all f_i are continuous and linear on the boundary. Let $u(x, y) = ax + by + cxy + d$.
 $u(x, y) = x - xy$.

[2] 3. Consider the following Sturm-Liouville problem $y'' + 3y' + \lambda y = 0$, $y(0) = y(2) = 0$. The weight function is:

(a) e^{3x} (b) e^{2x} (c) $3x$ (d) $2x$ (e) $\sin(2x)$

Solution: (a).

The integrating factor is $e^{\int 3dx}$.

- [7] 4. Consider the Laplace equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$, subject to the boundary condition $u(a, \theta) = f(\theta)$, where f is continuous, 2π -periodic, f' is piecewise continuous. Let $u(r, \theta)$ be bounded outside the circle $r = a$. Then the solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

$$a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad n \geq 0; \quad b_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta, \quad n \geq 1.$$

Question: Find the bounded solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 2$, subject to the boundary condition $u(2, \theta) = 4 \cos^2(\theta) - 3 \sin(3\theta)$.

Solution: $a = 2$. the solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Note that $4 \cos^2(\theta) = 2[\cos(2\theta) + 1]$. Thus

$$u(2, \theta) = 2 + 2 \cos(2\theta) - 3 \sin(3\theta).$$

Therefore

$$2 + 2 \cos(2\theta) - 3 \sin(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

which implies that

$a_0 = 4, 2 = 2^{-2}a_2, a_n = 0$ for $n \neq 0, 2; -3 = 2^{-3}b_3, b_n = 0$ for $n \neq 3$. Hence

$a_0 = 4, a_2 = 8, a_n = 0$ for $n \neq 0, 2; b_3 = -24, b_n = 0$ for $n \neq 3$. Hence

$$u(r, \theta) = 2 + 8r^{-2} \cos(2\theta) - 24r^{-3} \sin(3\theta).$$

- [7] 5. The solution of the heat equation $w_t = \alpha^2 w_{xx}$, $0 \leq x \leq L, t \geq 0$, subject to BCs: $w(0, t) = 0$, $w(L, t) = 0$ and IC: $w(x, 0) = f(x)$, is given by

$$w(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Question: Solve the following initial-boundary-value problem for $U(x, t)$:

$$\begin{aligned} U_t &= U_{xx} & 0 \leq x \leq 2, t \geq 0 & \text{PDE} \\ U(0, t) &= 1 \quad U(2, t) = 3 & t > 0 & \text{BC} \\ U(x, 0) &= f(x) = \begin{cases} x + 2, & 0 \leq x < 1, \\ x + 1, & 1 \leq x < 2, \end{cases} & & \text{IC} \end{aligned}$$

Solution: Here $\alpha = 1$, $L = 2$, $A = 1$, $B = 3$. Thus

$$v(x) = x + 1. \quad (1 \text{ point})$$

Let $w(x, t) = U(x, t) - v(x)$. Then

$$\begin{aligned} w_t &= w_{xx} & 0 \leq x \leq 2, t \geq 0 & \text{PDE} \\ w(0, t) &= w(2, t) = 0 & t > 0 & \text{BC} \\ w(x, 0) &= f(x) - v(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & 1 \leq x < 2, \end{cases} & & \text{IC} \end{aligned}$$

Thus, for $n = 1, 2, 3, \dots$,

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L [f(x) - v(x)] \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{2} \int_0^2 [f(x) - v(x)] \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \int_0^1 \sin\left(\frac{n\pi x}{2}\right) dx = \left[-\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_0^1 \\ &= \frac{2}{n\pi} [1 - \cos\left(\frac{n\pi}{2}\right)]. \end{aligned}$$

$$U(x, t) = x + 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - \cos\left(\frac{n\pi}{2}\right)] \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{2}\right)^2 t}.$$