MAT 2377 A
Introduction to Probability: Part 1

Definition : A *random experiment* is an experiment or a process for which the outcome cannot be predicted with certainty.

Definition : The *sample space* (denoted S) of a random experiment is the set of all possible outcomes. An element $x \in S$ is called an outcome.

Example 1:
Here are examples of random experiments, along with their corresponding sample spaces:

(a) Selection of a plastic component from a collection of 15 compliant pieces and 12 non-compliant pieces. Here is one possible sample space:

$$S = \{\text{piece } 1, \text{piece } 2, \ldots, \text{piece } 27\}$$

(pieces 1-15 are compliant, and pieces 16-27 are non-compliant.)

(b) Lifetime of an electronic component. Here is one possible sample space:

$$S = \{t \in \mathbb{R} \,:\, t \geq 0\} = [0, \infty).$$

(c) Number of calls to a communication system from 3:00pm-4:00pm. Here is one possible sample space:

$$S = \{0, 1, 2, 3, \ldots\}.$$

(e) The **ordered** selection of two tools **without replacement** from a box of three tools $\{A, B, C\}$. Here is one possible sample space:

$$S = \{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}.$$

(f) The **ordered** selection of two tools **with replacement** from a box of two tools. Here is one possible sample space:

$$S = \{(A, A), (A, B), (B, A), (B, B)\}.$$

Definition: An event E is a subset of the sample space S. We say that E has occurred if the observed outcome x is an element of E.
Notation: The notation $x \in E$ means "x is an element of the set E".

Remarks [A few special events]:

- We say that the sample space S is a *certain* event, since we know for sure that the observed outcome will be an element of S.

- $\emptyset$ is the *empty set*. We say that the empty set is an *impossible* event since the observed outcome can never be an element of the empty set. (we introduce the empty set for technical reasons)

Example 2: Consider the random experiments of Example 1.
(a) Let C be the event that the plastic component is compliant

$$C = \{\text{piece } 1, \text{piece } 2, \cdots, \text{piece } 15\}.$$

(recall that we said pieces 1-15 were compliant and pieces 16-27 were non-compliant)
(c) Recall that the event with the number of calls from 3:00pm-4:00pm with sample space $S = \{0, 1, 2, 3, \cdots\}$. Here are some possible events:

- Let A be the event that there are less than two calls

$$A = \{0, 1\}.$$

- Let B be the event that there are **more** than 8 call

$$B = \{9, 10, 11, \ldots\}.$$

- Let C be the event that there are **at most** 3 calls

$$C = \{0, 1, 2, 3\}.$$

- Let D be the event that there are **at least** 6 calls

$$D = \{6, 7, 8, 9, 10, \ldots\}.$$

Interpretation of a Probability:

Goal: To define a measure of the probability or the chances that an event E will occur.

Note We introduce three interpretations of the probability $P(E)$, that is, the probability that the event E occurs:

1. Subjective Probability;
2. Equally Likely Model;
3. Relative Frequency Model.

Subjective Probability: We associate a real number $P(E)$ between 0 and 1 in a subjective manner to the event E.

- numbers closer to 0 are interpreted as less likely;
- numbers closer to 1 are interpreted as more likely.

Here, $P(E)$ can be viewed as our personal belief regarding the probability of the outcome of the event E.

Example:

(a) Suppose we ask a fan of the Montreal Canadians or the Ottawa Senators what is the probability that his favorite team will win the president's cup in 2017 (the team that ends the season with the most points wins the cup).

Using his judgement and his knowledge of hockey, the fan might reply that there is a 45% chance of his favorite team winning the president's cup in 2017. We interpret the 45% like a belief that the event will happen.

Classical Approach: The first definition of a probability is found below. The model is called the equally likely model.

[**Equally Likely Model**]**:** Consider a random experiment with a finite sample space S such that each result has the same chance to occur. The probability that E will occur is

$$P(E) = \frac{N(E)}{N(S)} = \frac{\#\text{ favourable outcomes}}{\#\text{ total possible outcomes}}$$

where $N(E) = \#$ of outcomes in E.

Remark: We consider a **random selection** of an object among N objects as an experiment with equally likely outcomes.

Example: Consider the selection of a plastic component from a collection of 15 compliant pieces and et 12 non-compliant pieces. What is the probability that the selected piece will be compliant?

Assuming that each piece has the same chance of being selected, using the classical approach, we have

$$P(C) = \frac{N(C)}{N(S)} = \frac{15}{27} = 0.5556,$$

where C is the event that the selected piece is compliant.

Frequentist approach :

[**Relative Frequency Model**]**:** Consider a random experiment with a sample space S. We repeat the experiment n times. The probability that the event E will occur is

$$P(E) = \lim_{n\to\infty} \frac{f_n(E)}{n},$$

where $f_n(E)$ is the number of times (the frequency) that event E occurs among the n trials of the experiment.

Example :
Among the last 100,000 packets passing through a particular communication

channel, there were 500 corrupted packets. What is the probability that any given packet passing through the communication channel will be corrupted?

Solution: Using the frequentist approach, the probability that any given packet will be corrupted is (approximately) 500/100,000=0.005.

Enumerating Techniques

To answer more complicated probability problems, such as those involving a sequence of steps, we will need to develop techniques for counting.

Probability trees: If an experiment can be described by a sequence of k steps, then we can illustrate all the possible outcomes in the sample space by use of a tree. Each path in the tree represents an outcome.

Example 3: Our interest is a family with three children. What is the probability that this family has exactly two daughters?

Remark: While it is very easy to construct a tree, the tree can become very large rather quickly. We will therefore need other enumeration techniques.

Multiplication principle: If an experiment can be described by a sequence of k steps, and there are n_i ways to accomplish each step i, then

$$\# \text{ possible outcomes } = n_1 \times n_2 \times \cdots \times n_k.$$

Example 4: We first select one of three operator, one of two machines, and then we take one of five measurements using this machine. How many ways can we perform this experiment?

Solution: $3 \cdot 2 \cdot 5$

n Factorial ($n!$)

Definition:

Let n be a non-negative integer, that is $n = 0, 1, 2, \ldots$. We define n *factorial* by

$$n! = \begin{cases} n\,(n-1)\,(n-2) \times \cdots \times 1, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases}$$

Example 5: $0! = 1$; $1! = 1$; $2! = 2 \cdot 1 = 2$; $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$; $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Many experiments can be described by an arrangement or a selection of objects. Since we will frequently encounter these types of experiments, we will give them special names:

Definition: Consider n distinct objects. An arrangement of r of these objects is called a **permutation**.

Remark: For a permutation the order matters.

Notation:

P_r^n =# of different permutations of r objects chosen from a collection of n distinct objects.

Definition: Consider n distinct objects. A selection of r objects is called a **combination**.

Remark: For a combination the order does **not** matter.

Notation:

$C_r^n = \begin{pmatrix} n \\ r \end{pmatrix} =$ # of different combinations of size r that can be chosen from a collection of distinct n objects.

Example 6: Consider the set of objects $\{a, b, c, d\}$. Here is a list of all permutations of size 2 of these objets:

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.$$

So, $P_2^4 = 12$

And here is a list of their combinations of size 2:

$$ab, ac, ad, bc, bd, cd.$$

So $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$

Note: For a combination, what is matters is which objects we select, not the order in which we select them. So ab et ba represent the same combination.

Formulas:

$$P_r^n = n\,(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$$C_r^n = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!\,(n-r)!}$$

Proof: Consider the ordered selection of r objects among n objects. We have n choices for the first object, then $n-1$ choices for the second object, $n-2$ choices for the third object, and so on. For the last choice, there is the r'th object, we are left with $n-(r-1)$ object to choose from, so the number of choices will be $n-(r-1)=n-r+1$. Therefore, by the multiplication principle, we have

$$\begin{aligned} P_r^n &= n\,(n-1)\cdots(n-r+1) \\ &= n\,(n-1)\cdots(n-r+1)\frac{(n-r)(n-r-1)\cdots 1}{(n-r)(n-r-1)\cdots 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

We can also construct the above permutation in a different way. Choose r from the collection of n objects, irrespective of order. There are C_r^n possible unordered selections. Then, arrange our selection in order. We have r choices for the first object, then $r-1$ choices for the second object, $r-2$ choices for the third object, and so on. Therefore, there are $r\,(r-1)\,(r-2)\cdots 1 = r!$ different ways of arranging these r objects. Therefore, by the multiplication principle, we have

$$P_r^n = C_r^n\, r!.$$

But, $P_r^n = n!/(n-r)!$, so

$$C_r^n = \frac{n!/(n-r)!}{r!} = \frac{n!}{(n-r)!r!}.$$

Example 7: We have a group of 10 Engineers. We will select 3 engineers from this group to lead the committee on little projects, the committee on medium projects, and the committee on big projects, respectively (each committee must have exactly one leader, and no two committees can have the same leader). In how many different ways can we choose these three committee leaders?

Solution:

$$P_3^{10} = \frac{10!}{(10-3)!} = 10\cdot 9\cdot 8$$

Example 8: Consider a collection of 50 distincts articles, of which 3 are defective. We select 5 articles at random.

(a) In how many different ways can we choose the articles (ignoring the order in which they are selected)?
(b) What is the probability that there will be **exactly** 1 defective article from among the 5 chosen articles?
(c) What is the probability that there will be **at most** 1 defective article from among the 5 chosen articles?

Solution:
(a)

$$\binom{50}{5} = \frac{50!}{(50-5)! \cdot 5!}$$

(b) There are $\binom{3}{1}$ ways to choose the 1 defective article, and $\binom{47}{4}$ ways to choose the 4 non-defective articles. Therefore,

$$P(\text{exactly 1 defective article}) = \frac{\text{\# of ways to choose exactly 1 defective artilce}}{\text{\# of ways to choose 5 articles}} = \frac{\binom{3}{1}\binom{47}{4}}{\binom{50}{5}}$$

(c) What is the probability that there will be **at most** 1 defective article from among the 5 chosen articles?

$$P(\text{At most 1 defective article}) = P(\text{exactly 0 defective article}) + P(\text{exactly 1 defective article})$$

$$= \frac{\binom{3}{0}\binom{47}{5}}{\binom{50}{5}} + \frac{\binom{3}{1}\binom{47}{4}}{\binom{50}{5}}$$

Example 9: Suppose we have 12 students and that we want to divide the 12 students into 4 groups of size 3. Determine the number of different ways that we could distribute these 12 students into 4 groups.

Solution: There are $\binom{12}{3}$ ways of picking the first group, then $\binom{9}{3}$ ways of picking the second group, then $\binom{6}{3}$ ways of picking the third group, then $\binom{3}{3} = 1$ ways of picking the fourth group. But we don't care which group is picked "first," "second," "third," and "fourth," just who is in each group. Since there are 4! ways of numbering the groups, we get:

$$\#\text{ of ways} = \frac{\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{3}}{4!}$$