Partial Fraction Decomposition for Inverse Laplace Transform

Usually partial fractions method starts with *polynomial long division* in order to represent a fraction as a sum of a polynomial and an another fraction, where the degree of the polynomial in the numerator of the new fraction is less than the degree of the polynomial in its denominator:

$$\frac{s^3+1}{s^2+1} = s + \frac{-s+1}{s^2+1}.$$

We, however, *never* have to do this polynomial long division, when Partial Fraction Decomposition is applied to problems from Chapter 6.

Another important fact in Chapter 6 is that we use only the following *three* types of fractions:

$$1.\frac{s-a}{(s-a)^2+b^2},\quad 2.\frac{b}{(s-a)^2+b^2},\quad 3.\frac{1}{(s-a)^n},$$

because we know the corresponding Inverse Laplace transforms

$$1.\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos(bt),\ 2.\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = e^{at}\sin(bt), \tag{1}$$

$$3.\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at},\ \mathcal{L}^{-1}\left[\frac{1}{(s-a)^2}\right] = te^{at},\ \mathcal{L}^{-1}\left[\frac{1}{(s-a)^3}\right] = \frac{t^2}{2}e^{at}, \tag{2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-a)^4}\right] = \frac{t^3}{6}e^{at},\ \mathcal{L}^{-1}\left[\frac{1}{(s-a)^5}\right] = \frac{t^4}{24}e^{at}, \ldots \mathcal{L}^{-1}\left[\frac{1}{(s-a)^{n+1}}\right] = \frac{t^n}{n!}e^{at}$$

We will call fractions 1,2,3 as *standard fractions*. The Partial Fraction Decomposition for Inverse Laplace Transform is as follows.

Step 1 *Suitable decomposition.* The objective of this step is to give the correct format of the partial fraction decomposition for a given fraction.

Rules of suitable decomposition:

1. Numerator does not matter.
2. Number of standard fractions equals the degree of the denominator.
3. Number of undetermined constants equals the degree of the denominator.
4. All standard fractions involved should be different.

Simplest Scenario. When you solve a homogeneous equation $ay'' + by' + cy = 0$ you *always* have to solve

$$Y = \frac{??}{as^2+bs+c},$$

where I put ?? in the numerator (because by Rule 1 the numerator does not matter in Step 1), and in the denominator you will have the characteristic polynomial. Since the characteristic polynomial is quadratic you will need two different standard fractions (Rule 2 and 4) and two undetermined constants (Rule 3).

There are three cases here.

a) $as^2 + bs + c = 0$ has two distinct real roots. Example

$$\frac{??}{s^2-3s-4} = A\frac{1}{s-4} + B\frac{1}{s+1}.$$

where $s^2-3s-4=(s-4)(s+1)$. As in the example above, here the rule is that the two (different) standard fractions should be

$$\frac{1}{s-S_1},\ \frac{1}{s-S_2},$$

where S_1 and S_2 are the roots of $as^2+bs+c=0$.
Further examples

$$\frac{??}{s^2+s}=A\frac{1}{s}+B\frac{1}{s+1},\ \frac{??}{s^2-1}=A\frac{1}{s+1}+B\frac{1}{s-1}.$$

b) $as^2+bs+c=0$ has two repeated real roots. Example

$$\frac{??}{s^2-2s+1}=A\frac{1}{s-1}+B\frac{1}{(s-1)^2}.$$

where $s^2-2s+1=(s-1)^2$. As in the example above, here the rule is that the two (different) standard fractions should be

$$\frac{1}{s-S},\ \frac{1}{(s-S)^2},$$

where S is the repeated root of $as^2+bs+c=0$.
Further examples

$$\frac{??}{s^2}=A\frac{1}{s}+B\frac{1}{s^2},\ \frac{??}{s^2+6s+9}=A\frac{1}{s+3}+B\frac{1}{(s+3)^2}.$$

Note that for the cases a) and b) you will need to use the first formula in (2) for the Inverse Laplace Transform.
b) $as^2+bs+c=0$ has two complex roots. Example

$$\frac{??}{s^2-2s+5}=A\frac{s-1}{(s-1)^2+4}+B\frac{2}{(s-1)^2+4}.$$

where $s^2-2s+5=(s-1)^2+4$. As in the example above, here the rule is that the two (different) standard fractions should be

$$\frac{s-k}{(s-k)^2+m^2},\ \frac{b}{(s-k)^2+m^2},$$

where the denominator $(s-k)^2+m^2$ is obtained my completing the squares:

$$as^2+bs+c=a\left((s-k)^2+m^2\right).$$

Further examples

$$\frac{??}{4s^2+4s+5}=A\frac{s+1/2}{(s+1/2)^2+1}+B\frac{1}{(s+1/2)^2+1},\ \frac{??}{s^2+5}=A\frac{s}{s^2+5}+B\frac{\sqrt{5}}{s^2+5}.$$

More Complicated Scenario. When you solve a nonhomogeneous equation $ay''+by'+cy=g(t)$ you will have to deal with a fraction, which denominator is of degree 3 or higher. The same four rules of suitable decomposition still apply here:
1. Numerator does not matter.

2. Number of standard fractions equals the degree of the denominator.
3. Number of undetermined constants equals the degree of the denominator.
4. All standard fractions involved should be different.

The first preliminary step we have to take here is to decompose the denominator into a product standard polynomials. These standard polynomial are exactly of two types

$$a)\ (s-a)^n \ b) \text{ and } \left((s-k)^2+m^2\right).$$

Examples

$$(s-1)(s^2+4s+3)=(s-1)(s+1)(s+3),\ (s-1)(s^2+4s+5)=(s-1)((s+2)^2+1),$$

$$(s-1)(s^2+4s-5)=(s-1)(s-1)(s+5)=(s-1)^2(s+5).$$

Note that in the last example above we *have to* combine two terms $(s-1)$ into one $(s-1)^2$, so that all the involved standard polynomials have *distinct roots*.

The standard fractions that arise in this more complicated scenario are identical to the ones that we discussed in the simplest case with one exception. We may now need to use

$$\frac{1}{(s-S)^3},\ \frac{1}{(s-S)^4},\ \frac{1}{(s-S)^5},\ldots.$$

For example

$$\frac{??}{(s-3)^4}=A\frac{1}{s-3}+B\frac{1}{(s-3)^2}+C\frac{1}{(s-3)^3}+D\frac{1}{(s-3)^4}.$$

For everything else we just use the rule that we need to get the appropriate number of fractions for each standard polynomial in our denominator and just add them up together. For example, suppose you need to find a suitable form of a partial fractions decomposition for

$$\frac{??}{s^4((s+1)^2+1)(s^2+1)(s-1)^2(s^2-1)}$$

You first observe that there is one further decomposable fraction: $s^2-1=(s-1)(s+1)$. Therefore

$$\frac{??}{s^4((s+1)^2+1)(s^2+1)(s-1)^2(s^2-1)}=\frac{??}{s^4((s+1)^2+1)(s^2+1)(s-1)^3(s+1)},$$

and the denominator is the product of five standard fractions: s^4, $(s+1)^2+1$, s^2+1, $(s-1)^3$, $s+1$. For each one of them individually we know what we should do:

$$\frac{??}{s^4}=A\frac{1}{s}+B\frac{1}{s^2}+C\frac{1}{s^3}+D\frac{1}{s^4},$$

$$\frac{??}{(s+1)^2+1}=A\frac{s+1}{(s+1)^2+1}+B\frac{1}{(s+1)^2+1},$$

$$\frac{??}{s^2+1}=A\frac{s}{s^2+1}+B\frac{1}{s^2+1},$$

$$\frac{??}{(s-1)^3}=A\frac{1}{s-1}+B\frac{1}{(s-1)^2}+C\frac{1}{(s-1)^3},$$

$$\frac{??}{s+1}=A\frac{1}{s+1}.$$

Then, for the product $s^4((s+1)^2+1)(s^2+1)(s-1)^3(s+1)$, we should just add the individual suitable partial decompositions

$$\frac{??}{s^4((s+1)^2+1)(s^2+1)(s-1)^3(s+1)} = A\frac{1}{s} + B\frac{1}{s^2} + C\frac{1}{s^3} + D\frac{1}{s^4} + E\frac{s+1}{(s+1)^2+1}$$

$$+F\frac{1}{(s+1)^2+1} + G\frac{s}{s^2+1} + H\frac{1}{s^2+1} + I\frac{1}{s-1} + J\frac{1}{(s-1)^2} + K\frac{1}{(s-1)^3} + L\frac{1}{s+1}.$$

Step 2 *Evaluation of the undetermined constants.*

Here *numerator matters.* A first preliminary step here is to reduce the problem to an equality of polynomials. The bullet-proof way to do it is to multiply by the denominator of your given fraction in its standard form.

$$\frac{s^2+2s+3}{(s+1)(s^2+2s+2)} = A\frac{1}{s+1} + B\frac{s+1}{(s+1)^+1} + C\frac{1}{(s+1)^+1},$$

where $(s+1)(s^2+2s+2) = (s+1)((s+1)^2+1)$.

$$\frac{((s+1)((s+1)^2+1))\,(s^2+2s+3)}{(s+1)(s^2+2s+2)} = A\frac{(s+1)((s+1)^2+1)}{s+1}$$

$$+B\frac{(s+1)(s+1)((s+1)^2+1)}{(s+1)^+1} + C\frac{(s+1)((s+1)^2+1)}{(s+1)^+1},$$

$$\big((s+1)((s+1)^2+1)\big)\,(s^2+2s+3) = A((s+1)^2+1) + B(s+1)(s+1) + C(s+1).$$

Then you need to find the undetermined constants. You do that by finding linear equations for these constants. The number of equations should be equal to the number of unknowns. Apart from small tricks, there are two major methods: evaluation of your expression for some values of s, and comparison of coefficients for of the polynomials.

a) *Evaluation at some points* This method is very effective when the denominator has only *distinct real* roots.

$$\frac{s^2+1}{s(s+1)(s-1)} = A\frac{1}{s} + B\frac{1}{s+1} + C\frac{1}{s-1},$$

$$s^2+1 = A(s+1)(s-1) + Bs(s-1) + Cs(s+1),$$

$$s=0,\ 1=-A;\ \ s=1,\ 2=2C;\ \ s=-1,\ 2=2B.$$

$$A=-1,\ B=1,\ C=1.$$

Hence

$$\frac{s^2+1}{s(s+1)(s-1)} = \frac{1}{s} + \frac{1}{s+1} - \frac{1}{s-1}.$$

b) *Comparison of coefficients* This method is bullet-proof, but it is quite long.

$$\frac{s^2+1}{s(s+1)(s-1)} = A\frac{1}{s} + B\frac{1}{s+1} + C\frac{1}{s-1},$$

$$s^2+1 = A(s+1)(s-1) + Bs(s-1) + Cs(s+1),$$

$$s^2+1 = As^2 - A + Bs^2 - Bs + Cs^2 + Cs = (A+B+C)s^2 + (C-B)s - A,$$

On the right-hand side the coefficients in front of s^2, s, and 1 are 1, 0, and 1, respectively. On the left-hand side the coefficients in front of s^2, s, and 1 are $A+B+C$, $C-B$, and $-A$, respectively. Therefore

$$\begin{cases} 1 = A+B+C, \\ 0 = C-B, \\ 1 = -A, \end{cases}$$

Solving the above system we obtain

$$A = -1, \ B = 1, \ C = 1.$$

Hence

$$\frac{s^2+1}{s(s+1)(s-1)} = \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s-1}.$$

As you see, in the above example the evaluation method is somewhat better than the comparison method. Try to do the evaluation method for the next problem and compare with the following solution.

$$\frac{s^3}{(s^2+1)(s^2+4)} = A\frac{s}{s^2+1} + B\frac{1}{s^2+1} + C\frac{s}{s^2+4} + D\frac{2}{s^2+4}.$$

$$\begin{aligned} s^3 &= As(s^2+4) + B(s^2+4) + Cs(s^2+1) + D(s^2+1) \\ &= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D). \end{aligned}$$

$$\begin{cases} 1 = A+C, \\ 0 = B+D, \\ 0 = 4A+C, \\ 0 = 4B+D, \end{cases}$$

From the second and the fourth equation above we obtain $B = D = 0$. The first and the third equations give $A = -1/3$, $C = 4/3$.

$$\frac{s^3}{(s^2+1)(s^2+4)} = -\frac{1}{3}\frac{s}{s^2+1} + \frac{4}{3}\frac{s}{s^2+4}.$$