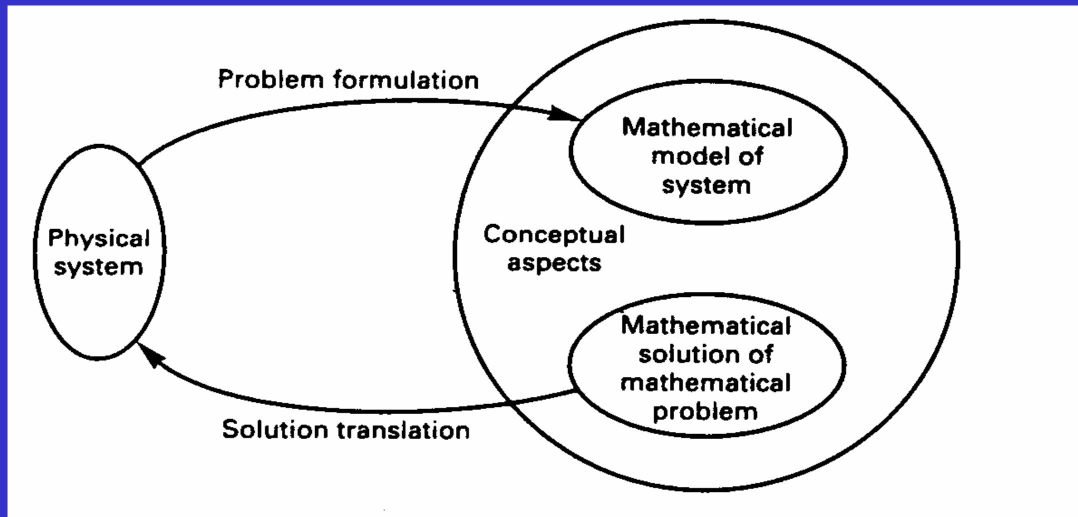
Mathematical Models of Physical Systems

- Why mathematical models of physical systems needed?
- Design of engineering systems by trying and error versus design by using mathematical models.
- Physical laws such as Newton’s second law of motion is a mathematical model.
- Mathematical model gives the mathematical relationships relating the output of a system to its input.

Mathematical Models of Physical Systems (continued)

- Control systems give desired output by controlling the input. Therefore control systems and mathematical modeling are inter-linked.



- Mathematical models of control systems are ordinary differential equations
- To deal with linear o.d.e, Laplace Transform is an efficient approach.

What is Laplace Transform?

- The Laplace transform of a function *f(t)* is defined as:

$$F(s) = \ell[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- The inverse Laplace transform is defined as:

$$f(t) = \ell^{-1}[F(s)] = \frac{1}{2j\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

where $j = \sqrt{-1}$ and the value of σ is determined by the singularities of *F(s)*.

$$s \equiv \frac{d}{dt}, \qquad \frac{1}{s} \equiv \int_0^t dt$$

Why Is Laplace Transform Useful ?

- Model a linear time-invariant analog system as an algebraic system (transfer function).
- In control theory, Laplace transform converts linear differential equations into algebraic equations.
- This is much easier to manipulate and analyze.

An Example

- The Laplace transform of e^{-at} can be obtained by:

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt = \frac{-e^{-(s+a)t}}{s+a} \Big|_0^{\infty} = \frac{1}{s+a}$$

Linearity property

- These are useful properties:

$$\ell[kf(t)] = k\ell[f(t)] = kF(s)$$

$$\ell[f_1(t) + f_2(t)] = \ell[f_1(t)] + \ell[f_2(t)] = F_1(s) + F_2(s)$$

Name	Time function f(t)	Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Unit ramp	t	$1/s^2$
nth-Order ramp	t^n	$n!/s^{n+1}$
Exponential	e^{-at}	$1/(s+a)$
nth-Order exponential	$t^n e^{-at}$	$n!/(s+a)^{n+1}$
Sine	$\sin(bt)$	$b/(s^2+b^2)$
Cosine	$\cos(bt)$	$s/(s^2+b^2)$
Damped sine	$e^{-at}\sin(bt)$	$b/((s+a)^2+b^2)$
Damped cosine	$e^{-at}\cos(bt)$	$(s+a)/((s+a)^2+b^2)$
Diverging sine	$t\sin(bt)$	$2bs/(s^2+b^2)^2$
Diverging cosine	$t\cos(bt)$	$(s^2-b^2)/(s^2+b^2)^2$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

table_02_02

Find the Laplace transform of $f(t)=5u(t)+3e^{-2t}$.

- Solution:

$$\ell[5u(t)] = 5\ell[u(t)] = \frac{5}{s}$$

$$\ell[3e^{-2t}] = 3\ell[e^{-2t}] = \frac{3}{s+2}$$

$$F(s) = \frac{5}{s} + \frac{3}{s+2} = \frac{8s+10}{s(s+2)}$$

Another Example:

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Partial Fraction Expansion

(Case 1: Roots of Denominator are Real and distinct)

Find the inverse Laplace transform of

$F(s)=5/(s^2+3s+2)$.

Solution:

$$F(s) = \frac{5}{s^2 + 3s + 2} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$k_1 = (s+1)F(s)|_{s=-1} = \frac{5}{s+2}|_{s=-1} = 5$$

$$k_2 = (s+2)F(s)|_{s=-2} = \frac{5}{s+1}|_{s=-2} = -5$$

$$F(s) = \frac{5}{s+1} + \frac{-5}{s+2}$$

So we have :

$$\ell^{-1}[F(s)] = f(t) = (5e^{-t} - 5e^{-2t})u(t)$$

Exercise: Do example 2.3 of the textbook

Laplace Transform solution of a differential equation

$$\frac{d^2 y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Case 2: Roots of the Denominator are Real and Repeated

$$F(s) = \frac{N(s)}{(s+p_1)^r (s+p_2)}$$ Note that roots are $-p_i$

$$F(s) = \frac{k_1}{(s+p_1)^r} + \frac{k_2}{(s+p_1)^{r-1}} + \ldots + \frac{k_r}{(s+p_1)} + \frac{k_{r+1}}{(s+p_2)}$$

$$Example\quad : F(s) = \frac{2s+3}{s^3+2s^2+s} = \frac{2s+3}{(s^2+2s+1)s}$$

$$= \frac{2s+3}{(s+1)^2 s}$$

$$= \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{k_3}{s}$$

$$Let\ \ F_1(s) = (s+p_1)^r F(s)$$

$$Then:\ \ k_i = \frac{1}{(i-1)!} \frac{d^{i-1} F_1(s)}{ds^{i-1}} \Big|_{s \to -p_i} \quad i = 1,2,\ldots,r;\ \ 0! = 1$$

Case 2: continue of the example *F(s)=(2s+3)/(s³+2s²+s).*

$$F(s) = \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{k_3}{s}$$

$$k_1 = (s+1)^2 F(s)\,|_{s=-1} = \frac{2s+3}{s}\Big|_{s=-1} = -1$$

$$k_2 = \frac{1}{(2-1)!}\frac{d}{ds}[(s+1)^2 F(s)]\Big|_{s=-1} = \frac{d}{ds}\left(\frac{2s+3}{s}\right)\Big|_{s=-1}$$

$$k_2 = \frac{2s-(2s+3)(1)}{s^2}\Big|_{s=-1} = \frac{-2-1}{1} = -3$$

$$k_3 = sF(s)\,|_{s=0} = \frac{2s+3}{(s+1)^2}\,|_{s=0} = 3$$

$$F(s) = \frac{-1}{(s+1)^2} + \frac{-3}{s+1} + \frac{3}{s}$$

$$f(t) = 3 - 3e^{-t} - te^{-t}$$

Case 3: Roots of the Denominator are Complex.

Example: F(s)=10/(s³+4s²+9s+10)

$$F(s) = \frac{10}{s^3 + 4s^2 + 9s + 10} = \frac{10}{(s+2)[(s+1)^2 + 2^2]}$$

$$F(s) = \frac{k_1}{s+2} + \frac{k_2}{s+1+j2} + \frac{k_2^*}{s+1-j2}$$

$$k_1 = (s+2)F(s)|_{s=-2} = \frac{10}{(s+1)^2 + 4}|_{s=-2} = 2$$

$$k_2 = (s+1+j2)F(s) = \frac{10}{(s+2)(s+1-j2)}|_{s=-1-j2}$$

$$k_2 = \frac{10}{(-1-j2+2)(-1-j2+1-j2)} = \frac{10}{(1-j2)(-j4)}$$

$$k_2 = \frac{10}{(2.236\ /-63.4^o)(4\ /-90^o)} = 1.118\ /153.4^o$$

$$f(t) = 2e^{-2t} + 2.23e^{-t}\cos(2t - 153.4^o)$$

$$f(t) = 2e^{-2t} - 2e^{-t}\cos(2t) + e^{-t}\sin(2t)$$

For the Tutorial next week, use Matlab to find the Inverse Laplace Transform for the case where the roots of the denominator are complex roots.

The Transfer Function

The *nth* - order differential equation :

$$\frac{\mathrm{d}^n c}{dt^n}+a_{n-1}\frac{d^{n-1}c}{dt^{n-1}}+\ldots+a_1\frac{dc}{dt}+a_0c=b_m\frac{\mathrm{d}^m r}{dt^m}+b_{m-1}\frac{d^{m-1}r}{dt^{m-1}}+\ldots+b_1\frac{dr}{dt}+b_0r$$

Laplace transform :

$$(s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0)C(s)=(b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s+b_0)R(s)$$

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0}$$

The characteristic equation is defined as :

$$s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0=0$$

The roots of this equation are called *poles* of the system.

The roots of the equation :

$$b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s+b_0=0$$

are called *zeros* of the system.

Transfer Function for a differential equation

Exercise: Do examples 2.4 and 2.5 on the Textbook.

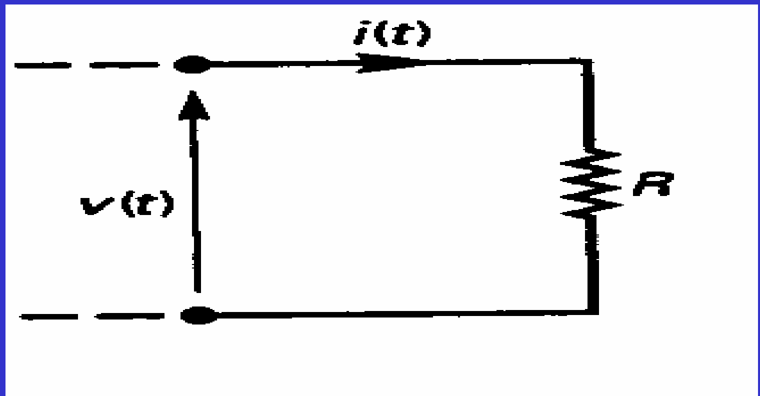
Models of Electrical Circuits

- Resistance circuit: $v(t) = i(t)\,R$

$$v(t) = i(t)\,R$$

$$V(s) = I(s)\,R$$

$$\frac{V(s)}{I(s)} = R$$

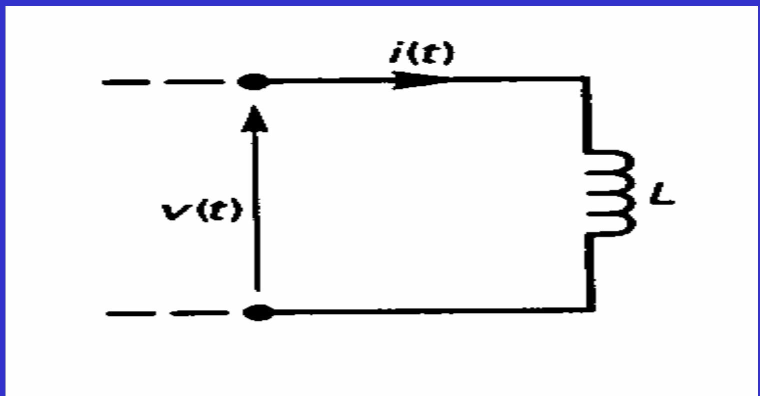


- Inductance circuit:

$$v(t) = L\frac{di(t)}{dt}$$

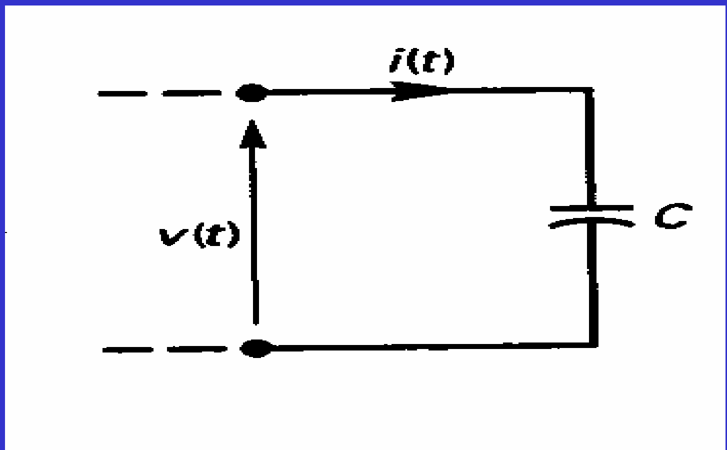
$$V(s) = L(sI(s) - i(0))$$

$$\frac{V(s)}{I(s)} = Ls \quad if \quad i(0) = 0$$



Models of Electrical Circuits

- Capacitance circuit:



$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

$$V(s) = \frac{1}{C}\frac{I(s)}{s} + \frac{v(0)}{s}$$

$$\frac{V(s)}{I(s)} = \frac{1}{Cs} \quad if \quad v(0) = 0$$

$$\frac{\ell(output)}{\ell(input)} = Transfer\ function$$

Transfer Functions for Electrical Networks

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\, d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

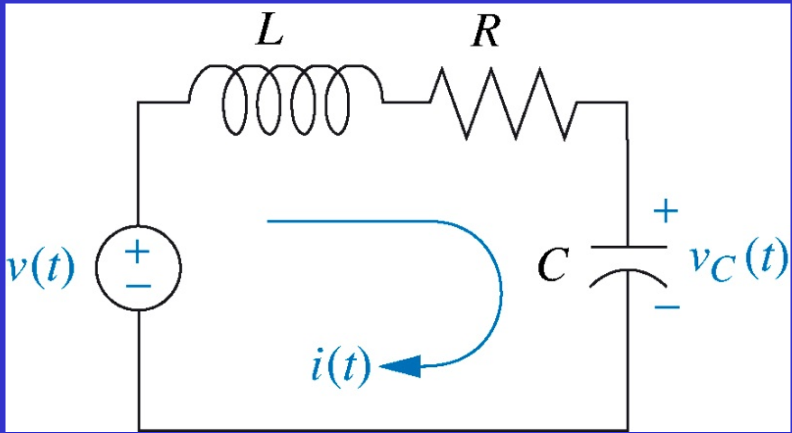
- Kirchhoff' s voltage law:

  The algebraic sum of voltages around any closed loop in an electrical circuit is zero.

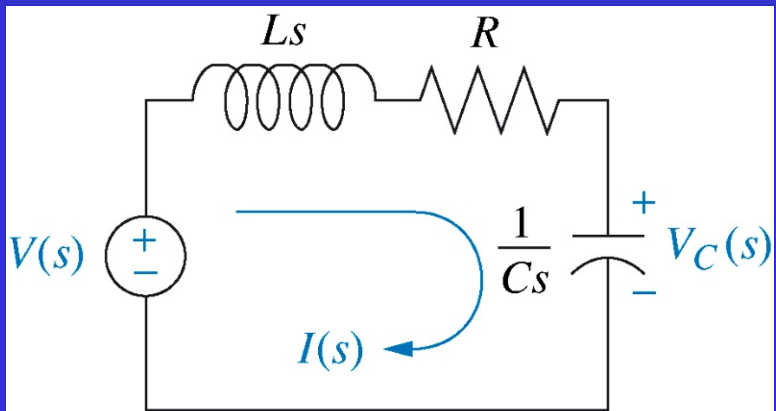
- Kirchhoff' s current law:

  The algebraic sum of currents into any junction in an electrical circuit is zero.

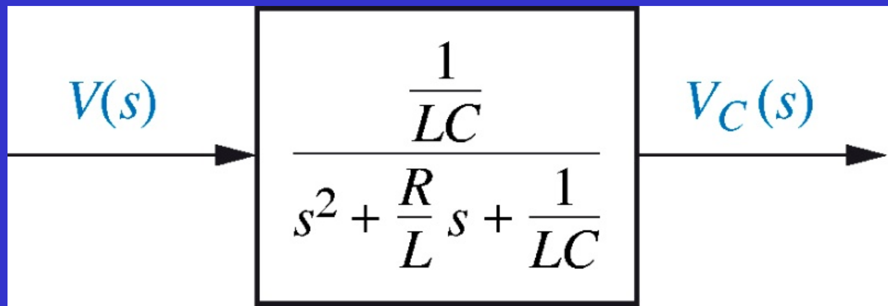
Electrical network (Example 2.6)



Original network



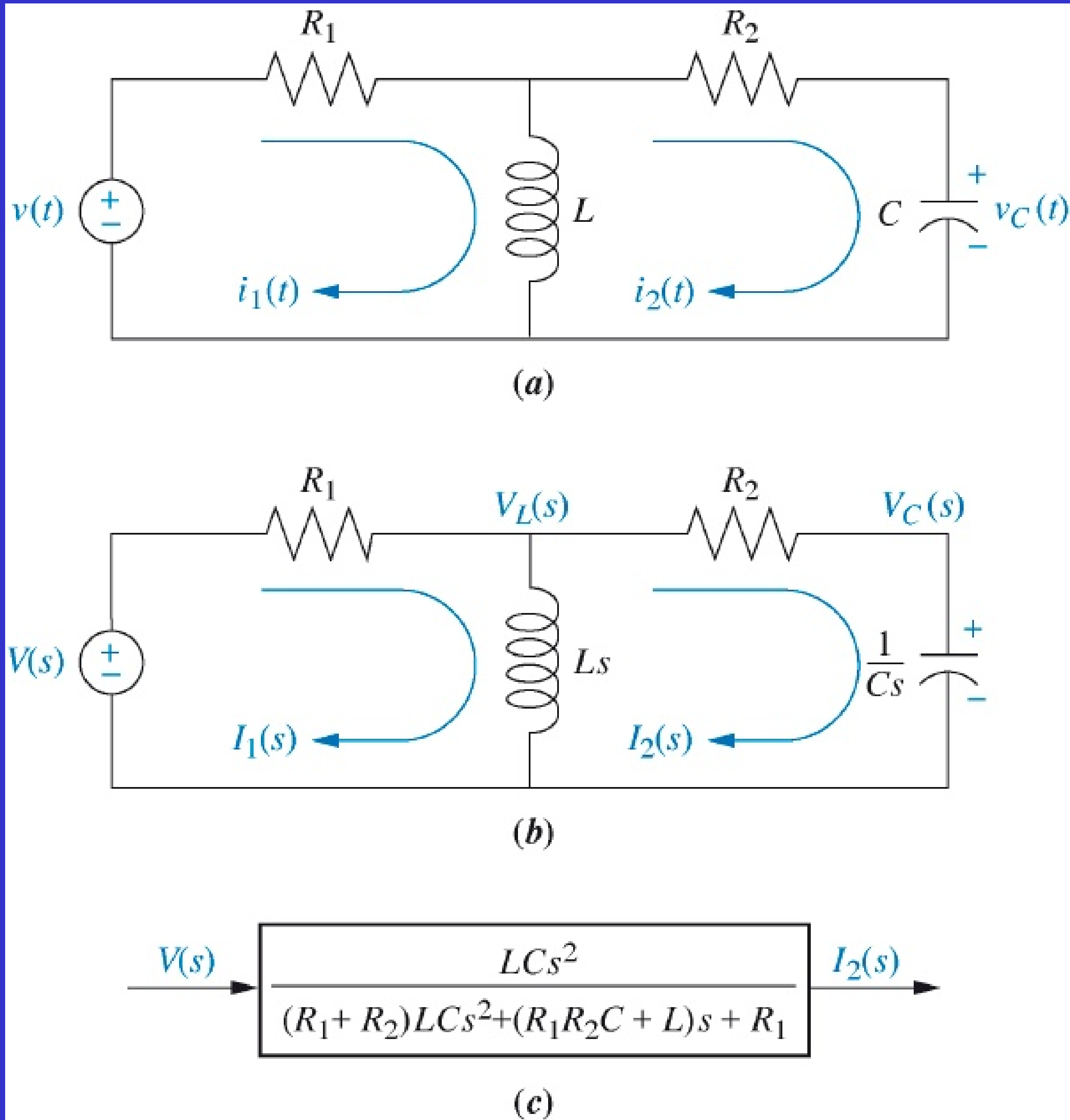
Laplace Transform network



Transfer Function network

- Exercise: Do example 2.10 of the textbook

- Excluding Operational Amplifiers



R_1
R_2
$v(t)$
L
C
$v_C(t)$
$i_1(t)$
$i_2(t)$
(a)
R_1
$V_L(s)$
R_2
$V_C(s)$
$V(s)$
Ls
$\frac{1}{Cs}$
$I_1(s)$
$I_2(s)$
(b)
$V(s)$
$\frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$
$I_2(s)$
(c)

fig_02_06

Models of Mechanical Systems

Mechanical translational systems.

- Newton's second law:

$$f(t) = Ma(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2}$$

- Device with friction (shock absorber):

$$f(t) = B[\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}]$$

B is damping coefficient.

- Translational system to be defined is a spring (Hooke's law):

$$f(t) = K[x_1(t) - x_2(t)]$$

K is spring coefficient

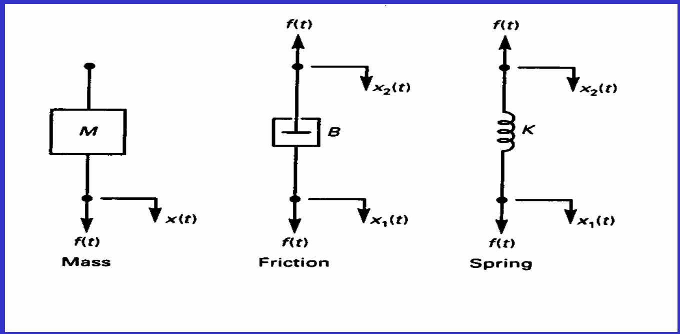


TABLE 2.4 **Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass**

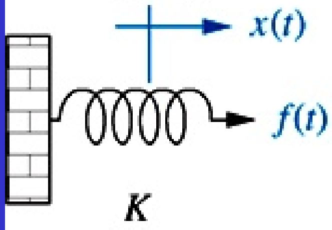
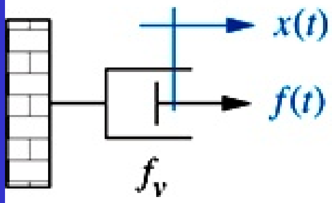
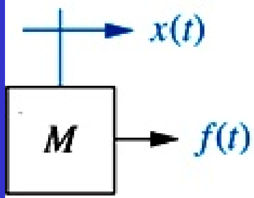
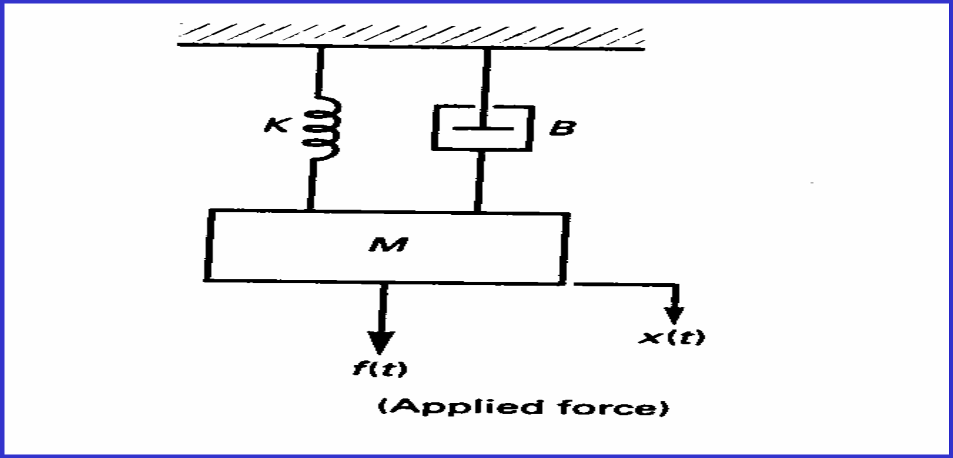
Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

table_02_04

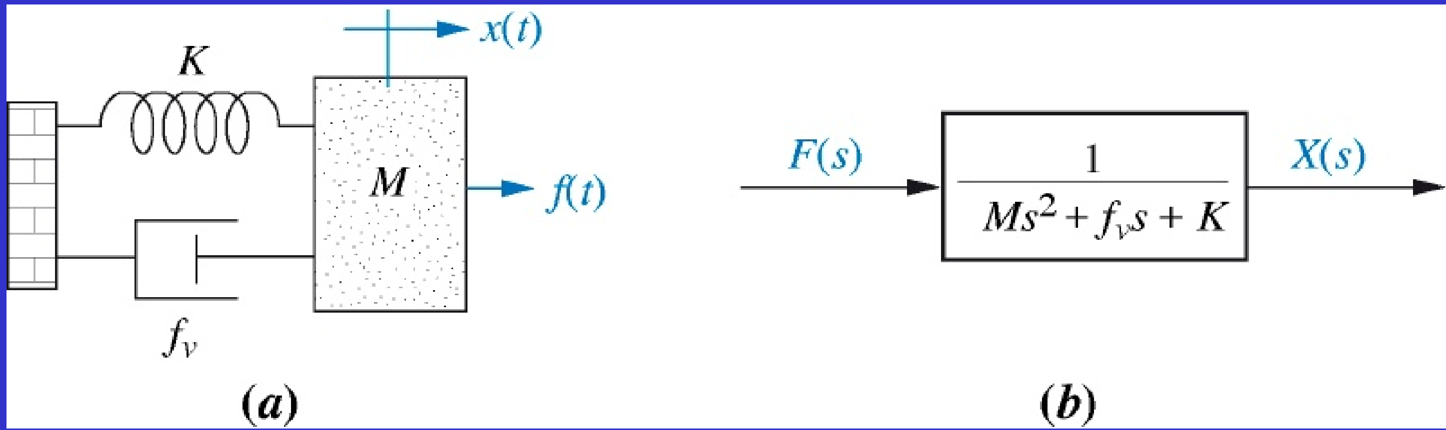
- Model of a mass-spring-damper system:

$$M\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+Kx(t)=f(t)$$

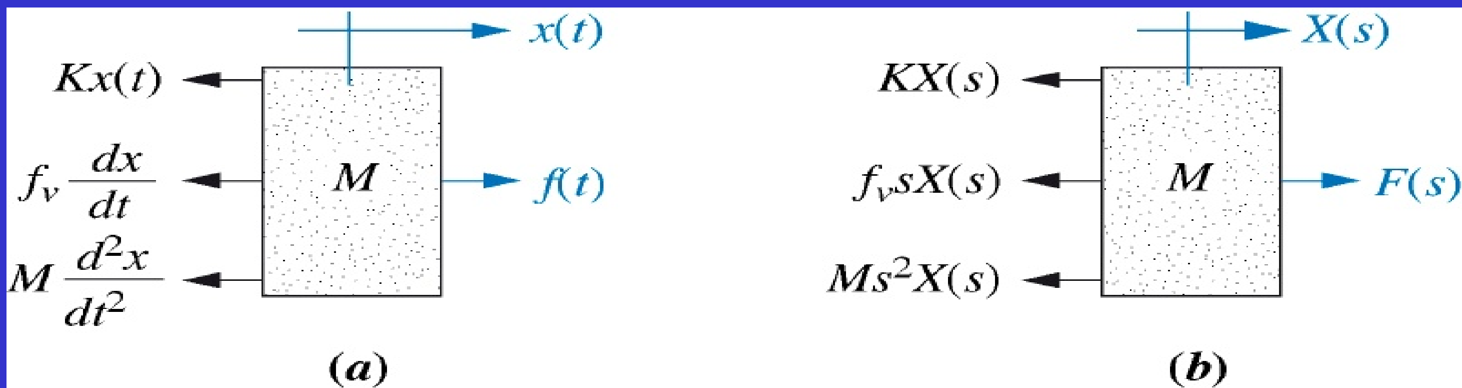


- Note that linear physical systems are modeled by linear differential equations for which linear components can be added together. See example of a mass-spring-damper system.

Transfer Functions for Mechanical Systems



$$M\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+Kx(t)=f(t) \Rightarrow Ms^2X(s)+BsX(s)+KX(s)=F(s)$$



Exercise: Do example 2.17 of the text

The idea is to set up the equations in the form :

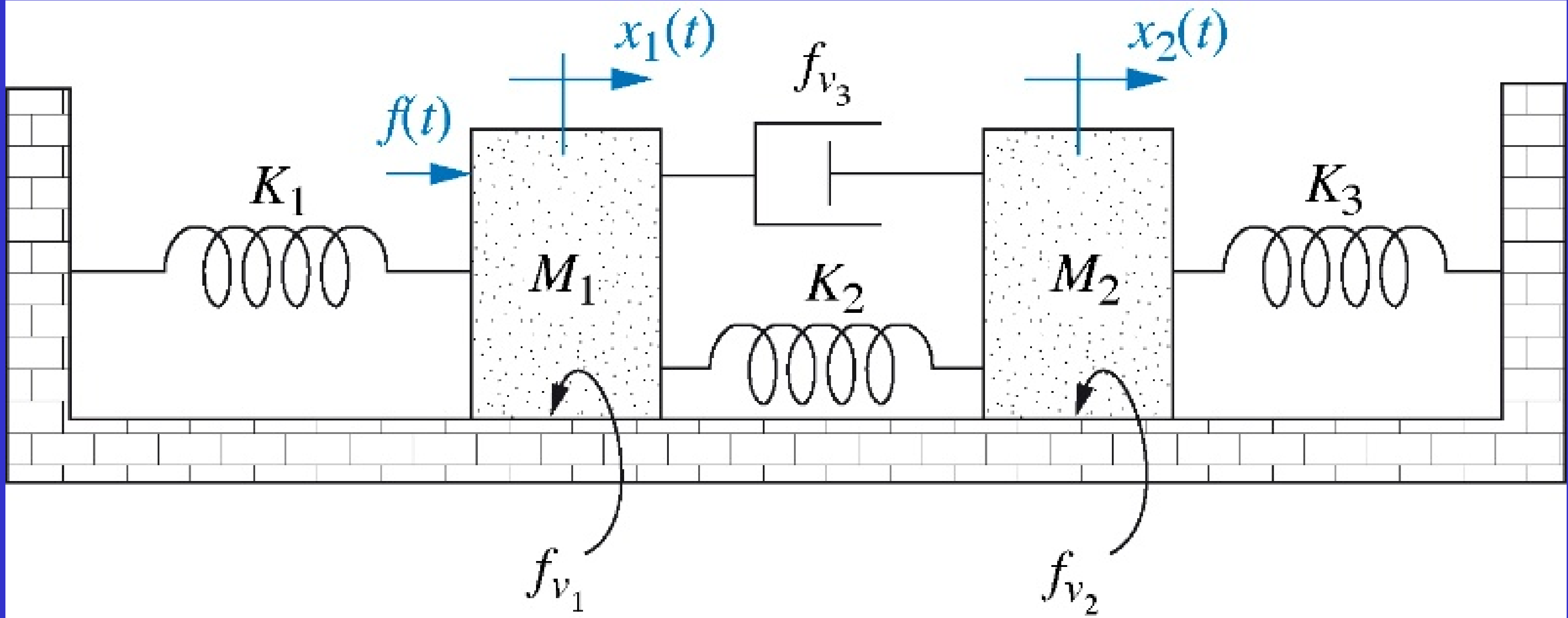
$$ax_1 + bx_2 = \alpha$$

$$cx_1 + dx_2 = \beta$$

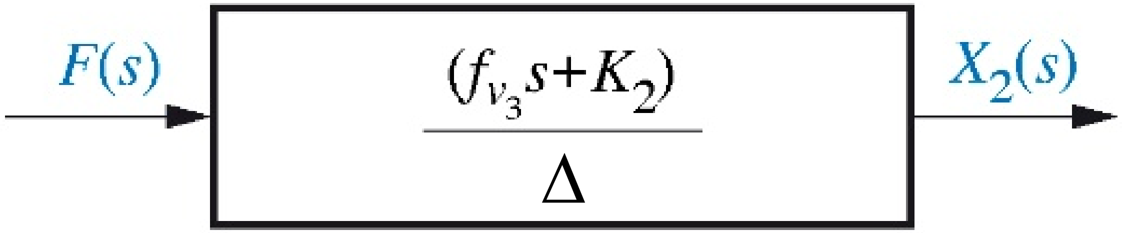
Using Cramer's rule, the solutions are then :

$$x_1 = \frac{\Delta_1}{\Delta}; \quad x_2 = \frac{\Delta_2}{\Delta}$$

where : $\Delta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \Delta_1 = \begin{bmatrix} \alpha & b \\ \beta & d \end{bmatrix}; \Delta_2 = \begin{bmatrix} a & \alpha \\ c & \beta \end{bmatrix}$



(*a*)



(*b*)

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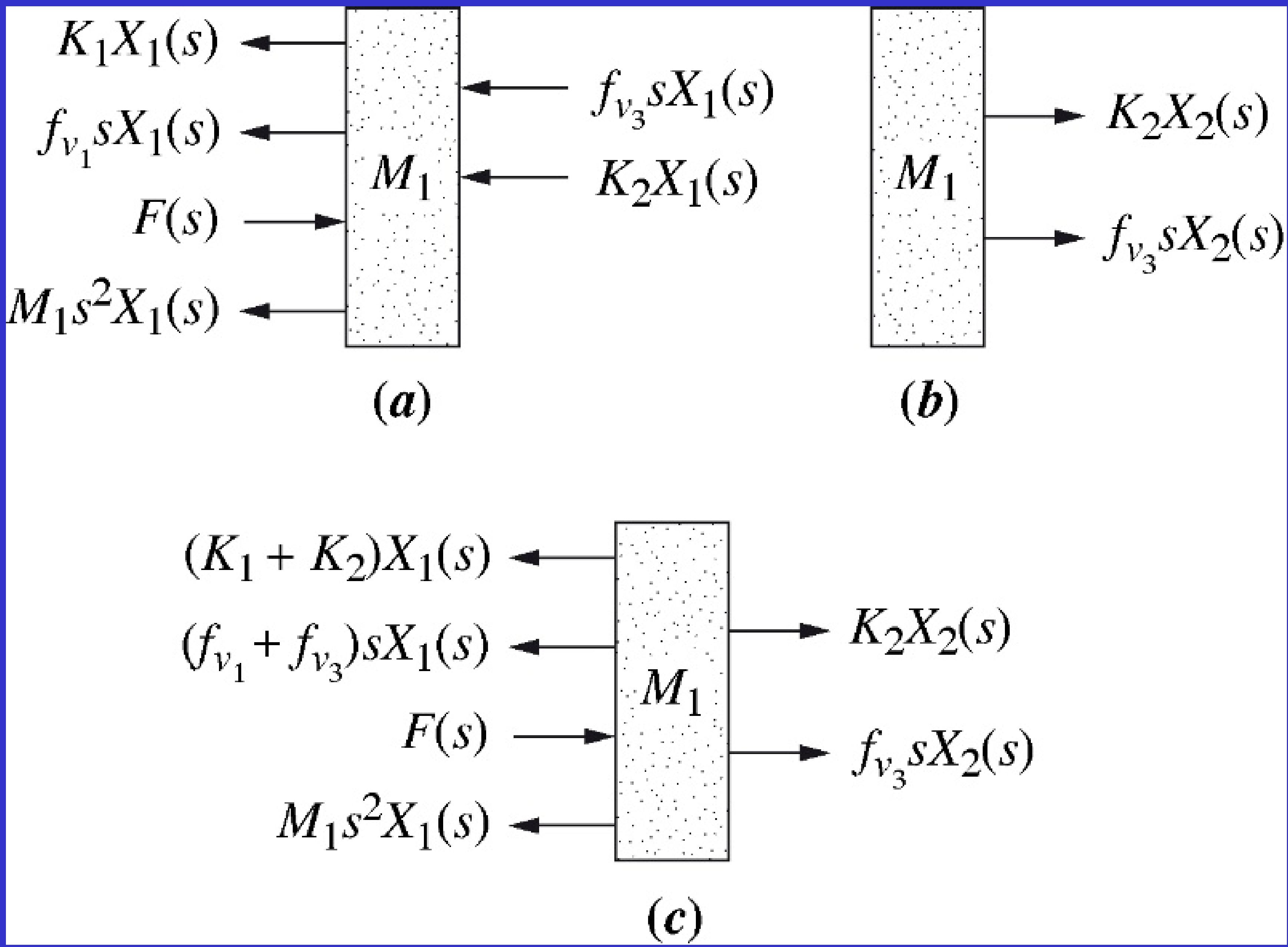


$K_1X_1(s)$
$f_{v_1}sX_1(s)$
$F(s)$
$M_1s^2X_1(s)$
M_1
$f_{v_3}sX_1(s)$
$K_2X_1(s)$
(a)
M_1
$K_2X_2(s)$
$f_{v_3}sX_2(s)$
(b)
$(K_1 + K_2)X_1(s)$
$(f_{v_1} + f_{v_3})sX_1(s)$
$F(s)$
$M_1s^2X_1(s)$
M_1
$K_2X_2(s)$
$f_{v_3}sX_2(s)$
(c)

fig_02_18

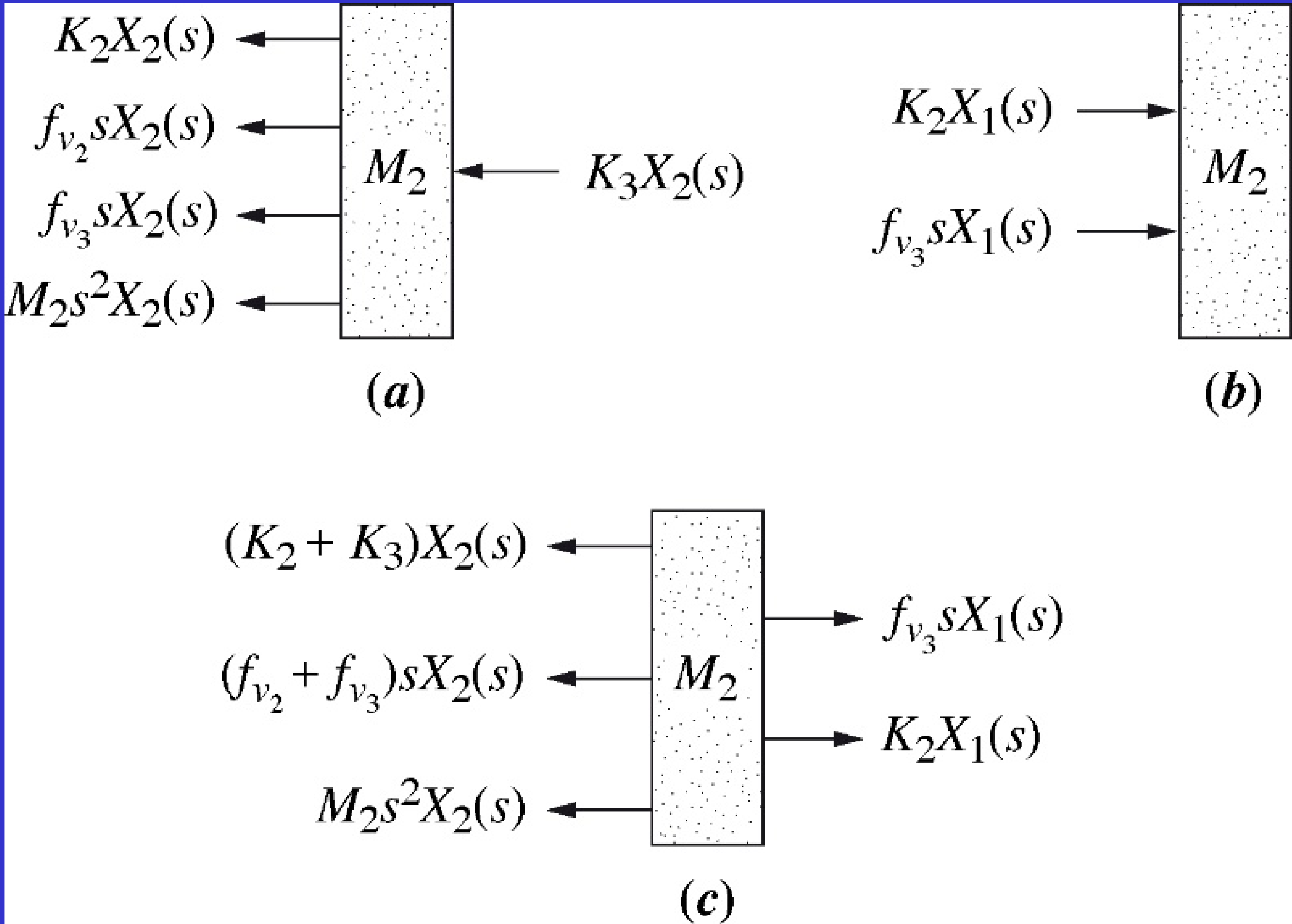
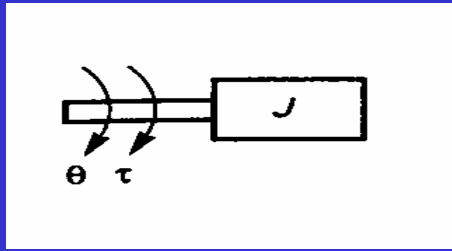

$K_2X_2(s)$
$f_{v_2}sX_2(s)$
$f_{v_3}sX_2(s)$
$M_2s^2X_2(s)$
M_2
$K_3X_2(s)$
(a)
$K_2X_1(s)$
$f_{v_3}sX_1(s)$
M_2
(b)
$(K_2 + K_3)X_2(s)$
$(f_{v_2} + f_{v_3})sX_2(s)$
$M_2s^2X_2(s)$
M_2
$f_{v_3}sX_1(s)$
$K_2X_1(s)$
(c)

fig_02_19

Mechanical Rotational Systems

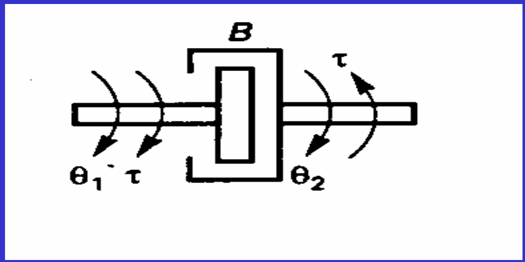
- Moment of inertia:

$$\tau(t) = J\frac{d^2\theta(t)}{dt^2} \qquad \tau(s) = Js^2\theta(s)$$



- Viscous friction:

$$\tau(t) = B(\frac{d\theta(t)}{dt}) \qquad \tau(s) = Bs\theta(s)$$



- Torsion:

$$\tau(t) = K[\theta(t)] \qquad \tau(s) = K[\theta(s)]$$

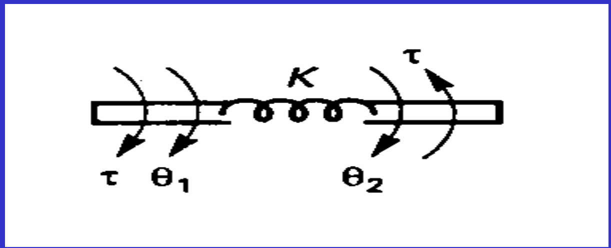


TABLE 2.5 **Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia**

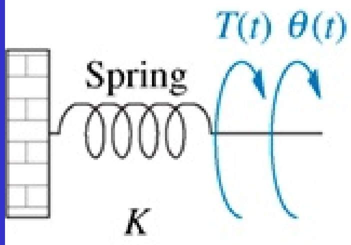
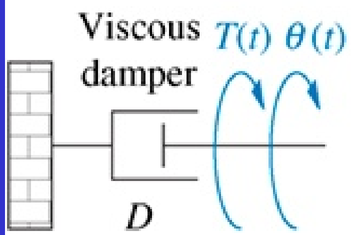
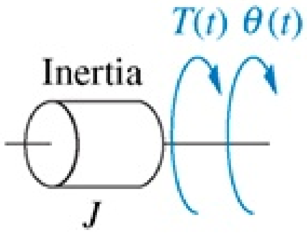
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

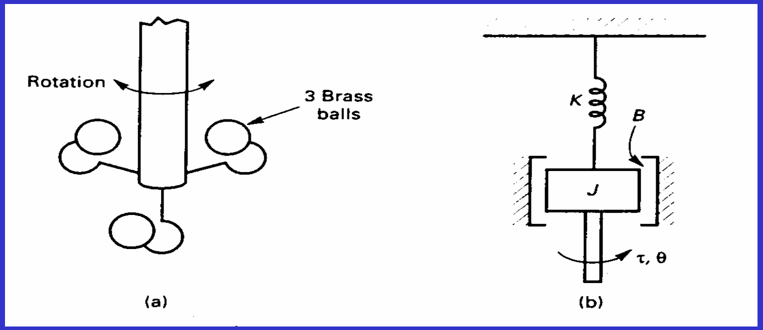
table_02_05

- Model of a torsional pendulum (pendulum in clocks inside glass dome):

  Moment of inertia of pendulum bob denoted by *J*

  Friction between the bob and air by *B*

  Elastance of the brass suspension strip by *K*

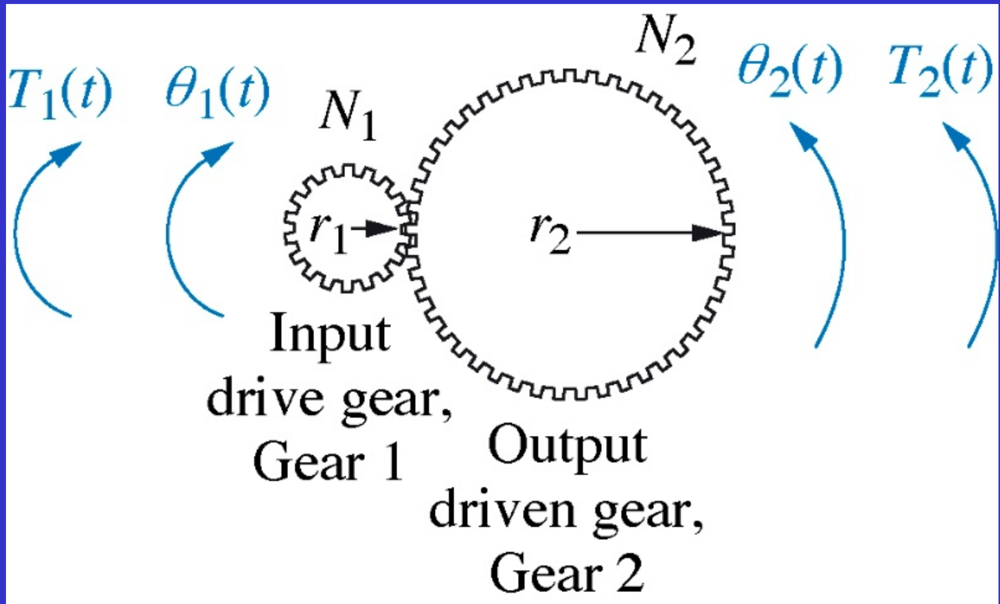


$$J\frac{d^2\theta(t)}{dt^2}+B\frac{d\theta(t)}{dt}+K\theta(t)=\tau(t)$$

$$Js^2\Theta(s)+Bs\Theta(s)+K\Theta(s)=\tau(s)$$

$$\frac{\Theta(s)}{\tau(s)}=\frac{1}{Js^2+Bs+K}=Transfer\ function$$

Transfer Functions for Systems with Gears

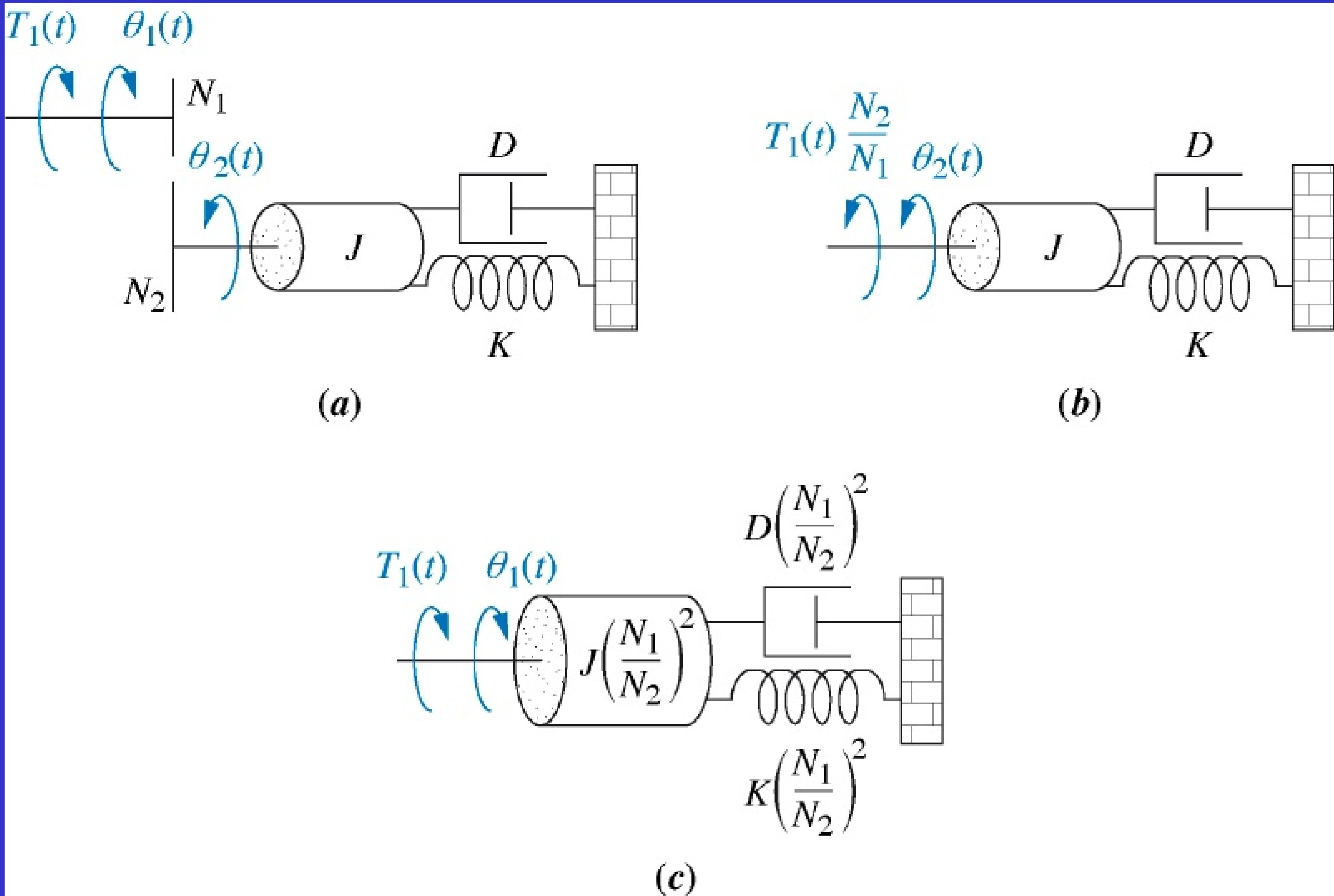


$$r_1\theta_1 = r_2\theta_2$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$$T_1\theta_1 = T_2\theta_2$$

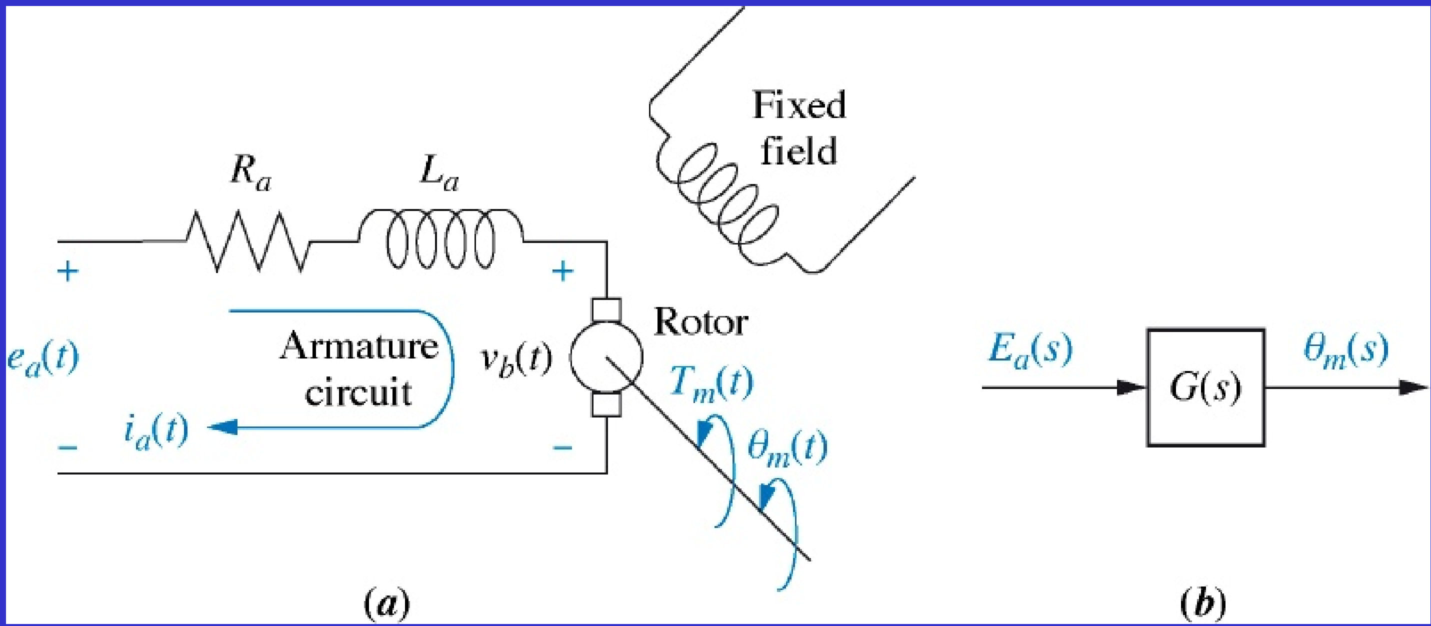
$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$



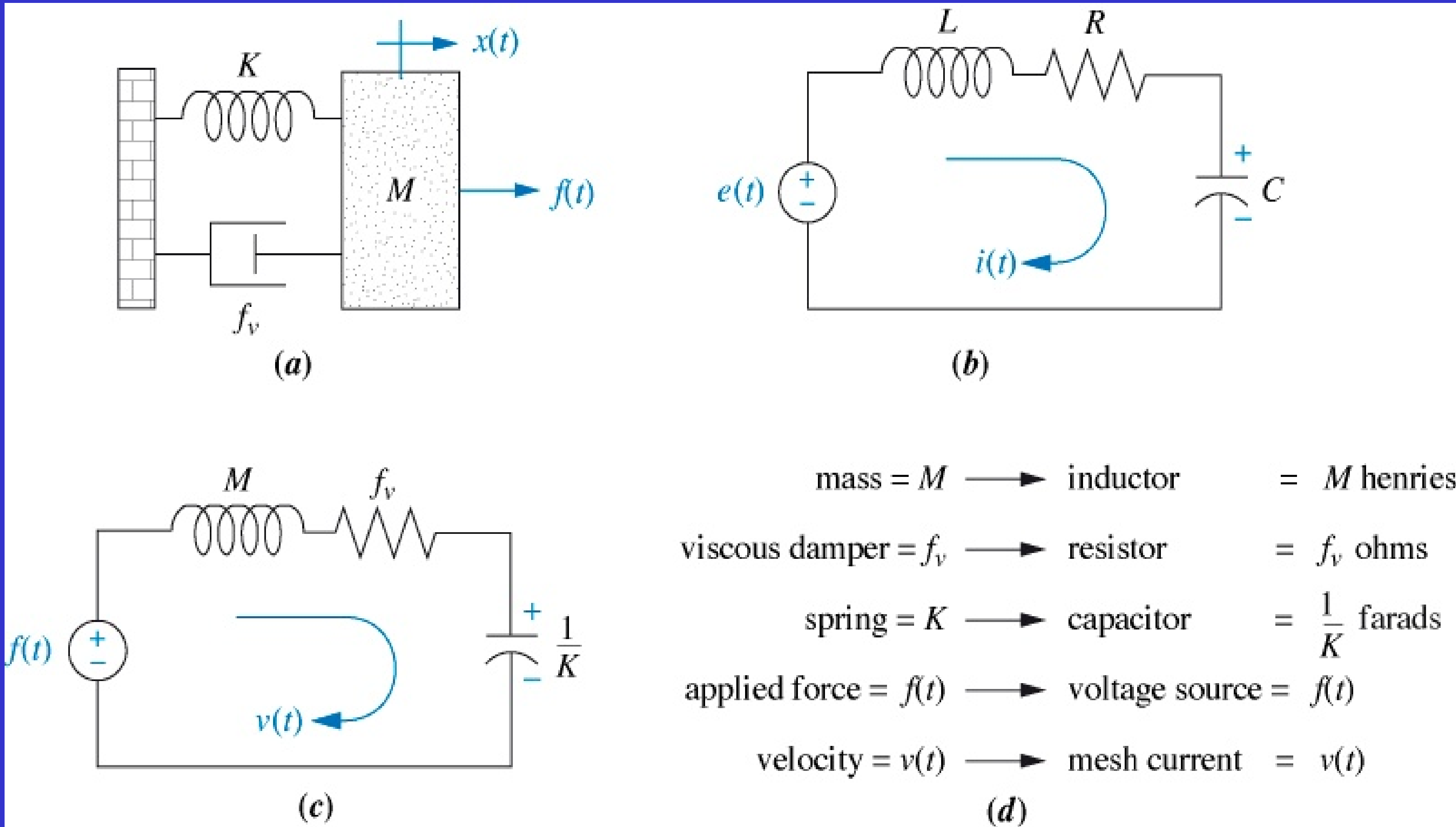
$T_1(t)$
$\theta_1(t)$
N_1
$\theta_2(t)$
N_2
D
J
K
(a)
$T_1(t)\frac{N_2}{N_1}$
$\theta_2(t)$
D
J
K
(b)
$T_1(t)$
$\theta_1(t)$
$D\left(\frac{N_1}{N_2}\right)^2$
$J\left(\frac{N_1}{N_2}\right)^2$
$K\left(\frac{N_1}{N_2}\right)^2$
(c)

fig_02_29

Electro-mechanical System Transfer Functions



Equivalence between Mechanical and Electrical Systems



- Differential equations as mathematical models of physical systems: similarity between mathematical models of electrical circuits and models of simple mechanical systems (see model of an RCL circuit and model of the mass-spring-damper system).

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = E(t)$$

$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Kirchhoff's and Newton's laws lead to mathematical models that describe the relationship between the input and output of dynamic system. One such model is the time invariant, *nth*-order differential equation:

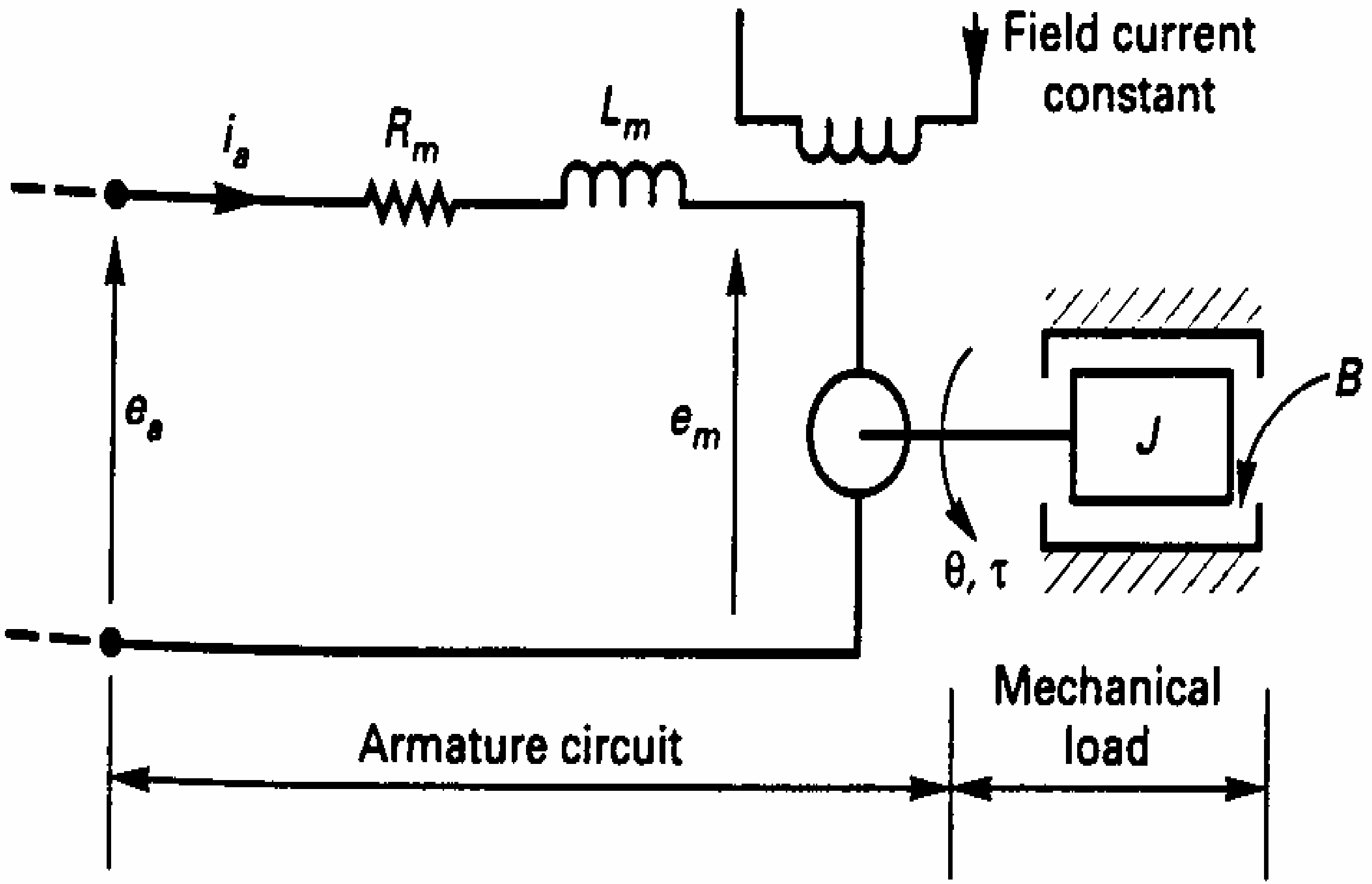
$$\frac{\mathrm{d}^{\mathrm{n}} c}{dt^{n}}+a_{n-1}\frac{d^{n-1}c}{dt^{n-1}}+\ldots+a_{1}\frac{dc}{dt}+a_{0}c=b_{m}\frac{\mathrm{d}^{\mathrm{m}} r}{dt^{m}}+b_{m-1}\frac{d^{m-1}r}{dt^{m-1}}+\ldots+b_{1}\frac{dr}{dt}+b_{0}r$$

where *c(t)* is the output, and *r(t)* is the input.

This leads to the concept of the Transfer Function which is the corner stone of the control theory. Transfer functions are defined from the Laplace Transform.

- Summary of through- and across-variables for physical systems:

System	Variable Through Element	Integrated Through Variable	Variable Across Element	Integrated Across Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical Translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, τ_{21}	



Field current
constant
i_a
R_m
L_m
e_a
e_m
J
B
θ, τ
Armature circuit
Mechanical
load

- Model of electromechanical systems.
- Model of a servomotor:

$$i_a(t) = \frac{1}{R_m}(e_a(t) - e_m(t))$$

$$e_m(t) = K_m \frac{d\theta}{dt}$$

$$\tau(t) = \frac{K_\tau}{R_m}(e_a(t) - e_m(t))$$

$$J\frac{d^2\theta(t)}{dt^2} = \tau(t) - B\frac{d\theta(t)}{dt}$$

$$J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} = k_1 e_a(t) - k_2\frac{d\theta}{dt}$$

- Model of temperature-control system:

$q_e(t) =$ heat flow given by electric heater

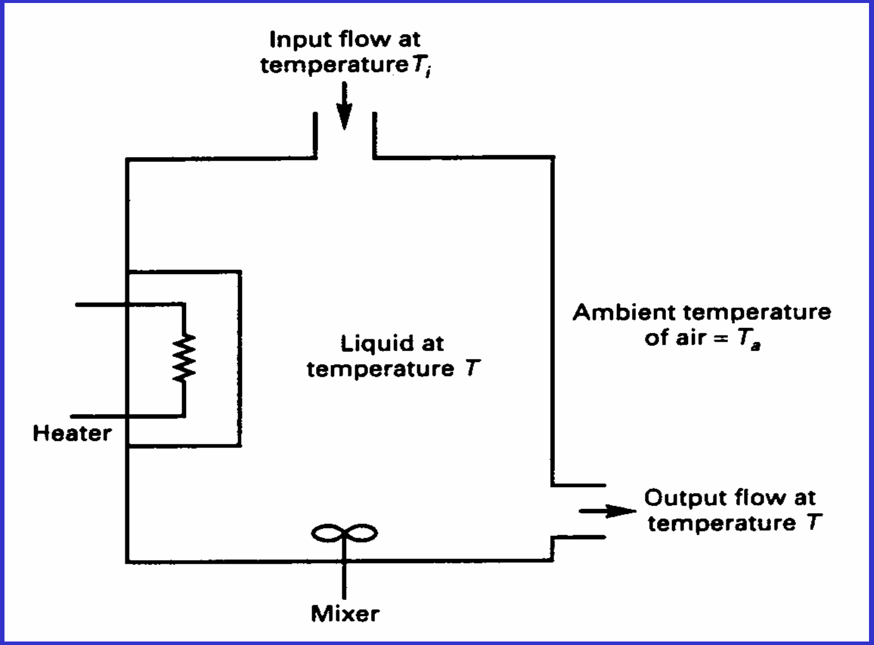
$q_i(t) =$ heat flow via liquid entering tank $= VHT_i$

$q_l(t) =$ heat flow into liquid $= C\, {dT}/{dt}$

$q_o(t) =$ heat flow via liquid leaving tank $= VHT$

$q_s(t) =$ heat flow (loss) via tank surface $= \dfrac{T - T_a}{R}$

$$q_e(t) + q_i(t) = q_l(t) + q_o(t) + q_s(t)$$



H = specific heat of liquid

C = thermal capacity

R = thermal resistance

V = volume of liquid

$$q_e + VHT_i = C\frac{dT}{dt} + VHT + \frac{T - T_a}{R}$$