University of Ottawa
Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
Professor: Aziz Khanchi

First Midterm Test
February 5, 2010

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Student # 5583638 DGD (1–4) ______

Instructions:

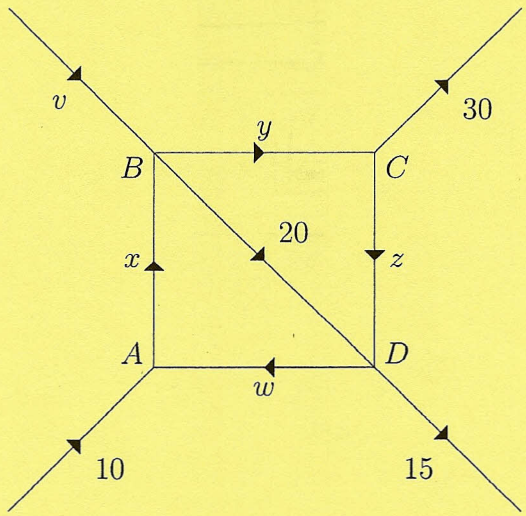
(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	5	5	4	4	4	25
Grade	2	5	2	4	4	2	19

1. [**3 pts**] Write down a system of equations describing the following traffic flow problem. The letters A through D label intersections and numerical values indicate flow in cars per minute. The arrows indicate the direction of flow (all roads are one-way). Include all relevant equations. You do **not** need to solve the system.



input = output

nodes (intersection)	input	output
A	$10+w$	x
B	$v+x$	$y+20$
C	y	$30+z$
D	$20+z$	$w+15$

$10+v = 30+15$

linear system.

$$\begin{cases} 10+w=x \\ v+x=y+20 \\ y=30+z \\ 20+z=w+15 \end{cases} \Rightarrow \begin{cases} 10+w-x=0 \\ v+x-y-20=0 \\ y-30-z=0 \\ 5+z-w=0 \end{cases}$$

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2. **[5 pts]** Find the general solution to the matrix-vector equation

$$\begin{bmatrix} 1 & -1 & 2 & 3 & -2 \\ 2 & -2 & 6 & 6 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & -2 \\ 2 & -2 & 6 & 6 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \vec{b} = ? = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 & -2 \\ 2 & -2 & 6 & 6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

convert to augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -2 & 3 \\ 2 & -2 & 6 & 6 & 4 & 6 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -2 & 3 \\ 0 & 0 & 2 & 0 & 8 & 0 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & -10 & 3 \\ 0 & 0 & 2 & 0 & 8 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & -10 & 3 \\ 0 & 0 & 1 & 0 & 4 & 0 \end{array}\right]$$

x_1 & x_3 are basic

x_2, x_4, x_5 are free

general solution

$$\begin{cases} x_1 - x_2 + 3x_4 - 10x_5 = 3 \\ x_2 \text{ is free} \\ x_3 + 4x_5 = 0 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases} \qquad \begin{cases} x_1 = x_2 - 3x_4 + 10x_5 \\ x_2 \text{ is free} \\ x_3 = -4x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$$

$$(x_2 - 3x_4 + 10x_5,\ x_2,\ -4x_5,\ x_4,\ x_5)$$

$$= \begin{bmatrix} x_2 - 3x_4 + 10x_5 \\ x_2 \\ -4x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

3. **[5 pts]** Is the following linear system consistent or inconsistent? If it is consistent, find the general solution.

$$x_1 + 2x_3 + 7x_4 + 5x_5 = x_2 + 1$$
$$-x_1 + 2x_2 = x_2 + x_4 + x_5 - 3$$
$$x_1 - x_2 + x_3 + 5x_4 + 6x_5 - 5 = -x_1 + x_2 + 2x_5$$

rewrite equation to have constan only on one side.

$$x_1 - x_2 + 2x_3 + 7x_4 + 5x_5 = 1$$
$$-x_1 + x_2 \quad - x_4 - x_5 = -3$$
$$-2x_2 + x_3 + 5x_4 + 4x_5 = 5$$

make into angumented

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 7 & 5 & 1 \\ -1 & 1 & 0 & -4 & -5 & -3 \\ 2 & -2 & 1 & 5 & 4 & 5 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 7 & 5 & 1 \\ 0 & 0 & 2 & 3 & 0 & -2 \\ 0 & -2 & 1 & 5 & 4 & 5 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 7 & 5 & 1 \\ 0 & -2 & 1 & 5 & 4 & 5 \\ 0 & 0 & 2 & 3 & 0 & -2 \end{array}\right] \xrightarrow{R_2 + (-2)R_1 \to R_1} \left[\begin{array}{ccccc|c} -2 & 0 & -3 & -9 & -6 & 3 \\ 0 & -2 & 1 & 5 & 4 & 5 \\ 0 & 0 & 2 & 3 & 0 & -2 \end{array}\right] \xrightarrow[\substack{-\frac{1}{2}R_2 \\ \frac{1}{2}R_3}]{-\frac{1}{2}R_1}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{2} & \frac{9}{2} & 3 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} & -2 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3 + R_2 \to R_2]{-\frac{3}{2}R_3 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 9/4 & 3 & -3 \\ 0 & 1 & 0 & -7/4 & -2 & -2 \\ 0 & 0 & 1 & 3/2 & 0 & 1 \end{array}\right]$$

the linear system <u>is Consistent</u> since there are no row with $[0 \cdots 0 | b]$ $b \neq 0$

x_1, x_2 x_3 are basic

x_4, x_5 are free

$$\begin{cases} x_1 = -3 - 9/4\, x_4 - 3x_5 \\ x_2 = -2 + 7/4\, x_4 - 2x_5 \\ x_3 = -1 - 3/2\, x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$$

$(-3 - 9/4x_4 - 3x_5,\ -2 + 7/4 - 2x_5,\ -1 - \frac{3}{2}x_4 - 2x_5,\ x_4,\ x_5)$

4. [**4 pts**] Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ -2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -1 \\ 4 \\ 7 \\ 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}.$$

Is $\vec{b}$ in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

aug. matrix

$$\left[\begin{array}{ccc|c} 0 & 3 & -1 & 2 \\ 1 & -2 & 4 & 1 \\ 2 & -1 & 7 & 3 \\ 1 & -2 & 4 & 1 \end{array}\right] \xrightarrow{-R_2+R_4 \to R_4} \left[\begin{array}{ccc|c} 0 & 3 & -1 & 2 \\ 1 & -2 & 4 & 1 \\ 2 & -1 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 2 & -1 & 7 & 3 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

the matrix is inconsistent $([0 \dots 0 | b]\ b \neq 0)$ there fore $\vec{b}$ is not in the span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

5. [**4 pts**] Determine if the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has infinitely many solutions or only one solution.

this is a homogeneous Vector eq. & is trivial. ∴ if there is at least 1 free variable it has infinite many soln. if not it only has 1 soln

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & 8 & 0 \\ 3 & 1 & 11 & 0 \end{array}\right] \xrightarrow[-3R_1 - R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right] \xrightarrow[\frac{1}{4}R_3]{\frac{1}{3}R_2}$$

$$\left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_1 + R_2 \to R_1]{-R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 are basic

x_3 is free

∴ Since x_3 is free then the vector eq. has infinite many soln'

G.S

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \end{cases} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

6. [**4 pts**] Suppose a linear system has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+10 & q+4 \end{array}\right]$$

for some real numbers p and q. For which values of p and q does the system have:

(a) No solution?
(b) Exactly one solution?
(c) Infinitely many solutions?

Your answers should include all possibilities for the values of p and q. Write your final answer in the spaces at the bottom of the page.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+10 & q+4 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & p+1 & q-2 \end{array}\right]$$

✓

$$\begin{cases} x_1 + 3x_2 = 2 \\ p+1x_2 = q-2 \end{cases} \qquad \begin{cases} x_1 = -3x_2 - 2 \\ x_2 = \dfrac{q-2}{p+1} \end{cases}$$

a) No soln if
$P = -1$
$q \in \mathbb{R}$

b) Exactly 1 soln
$-1 > P > -1$
$q = 2$

c) infinite many soln
$-1 > P > 1$
$q \in \mathbb{R}$

Final Answer: (a) $P = -1$ ~~$q \in \mathbb{R}$~~

(b) ~~$q = 2$~~ ~~$-1 > P > -1$~~

(c) ~~$-1 > P > 1$~~ ~~$q \in \mathbb{R}$~~