Your submission **must be created using Microsoft Word, Google Docs, or LaTeX**.

Your submission **must be saved as a single "pdf"** document and **have the name "a4.lastname.firstname.pdf"**

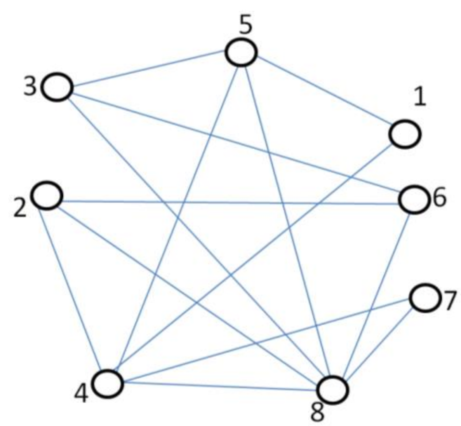
Do not compress your submission into a "zip" file.

Late assignments will not be accepted and will receive a mark of 0.

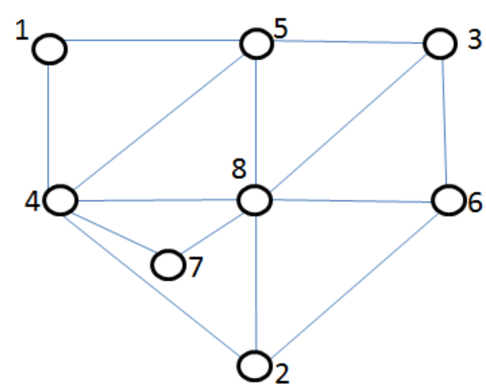
Submissions **written by hand**, **compressed into an archive**, or submitted in the **wrong format** (i.e., are not "pdf" documents) **will receive a mark of 0.**

The due date for this assignment is March 25th, 2017, by 11:30pm.

1. Can a planar representation be created for the following graph? If yes, create it (i.e. draw and label each node in the planar graph).



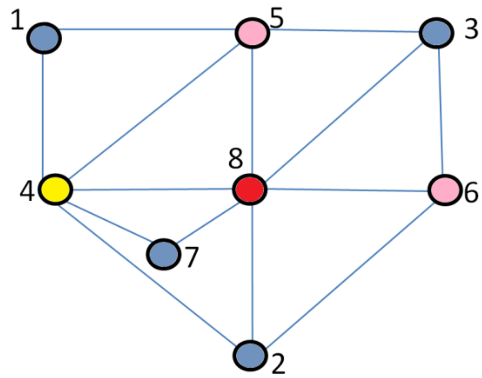
Possible solution:



2. Provide a valid colouring for the graph in Question 1 that uses the minimum number of colours. Remember that a valid colouring is one that does not assign the same colour to two adjacent vertices. Once you have found this colouring, prove that you

have used the minimum number of colours. You are not permitted to reference any theorems for answering this question. Use plain English to explain your solution.

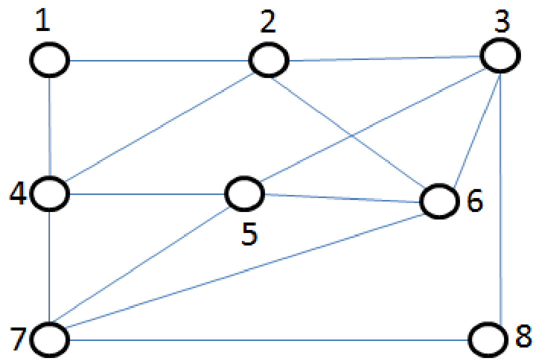
Possible Solution: Minimum number of colours: 4.



Example of an explanation:

This graph contains subgraph G' = (V', E') where V' = {2,3,4,5,6,8} and E' = {{2,4}, {2,6}, {2,8},{3,5},{3,6},{3,8},{4,5},{4,8}, {5,8}, {6,8}}. Any valid colouring of the original graph would also need to be valid over this subgraph. Assume that a valid 3-colouring of this graph exists, with three colours named x, y, and z. Vertex 8 must have a colour that is unique to all other vertices because it is adjacent to all of them. We can assign colour x to vertex 8. If vertex 2 is coloured using colour y, then vertex 4 must be coloured using colour z because 1 and 2 are adjacent. If there is a valid 3-colouring then vertices 5 and 6 would have to be coloured using either x, y, or z. However, if they are coloured x then there is a contradiction because 5 and 6 are adjacent to 8 because of the edges {5, 8} and {6,8}. Since 4 and 5 are adjacent, 5 cannot be coloured z, and since 6 and 2 are adjacent, 6 cannot be coloured y.
From here, we have two options.
Option 1 (not shown here): 5 can be coloured y and 6 can be coloured z (no contradictions here). However, since 3 is adjacent to 5, 6, and 8, with colours x, y, and z, then it must not have any of the three colours. This contradiction indicates that a valid 3-colouring does not exist for G', entailing that the graph cannot be coloured in less than 3 colours.
Option 2 (shown here): Alternatively, we can give 5 and 6 the fourth colour, leaving 3 the availability of being coloured y or z. Again, a valid 3-colouring does not exist for G', entailing that the original graph cannot be coloured in less than 3 colours either.
The minimum number of colours that can be used to colour the original graph cannot be less than the number required for subgraph G', so the minimum number of colours must be 4.

Use the following graph for 3 and 4.



3. An Euler path is a path that uses every **edge** of a graph exactly once, that starts at one node and ends at a different node. An Euler cycle, is an Euler path that starts and ends at the same node. Does the following graph have an Euler path? Does it have an Euler cycle? If it does, then list the edges in the order they are crossed, and if it does not then explain why.

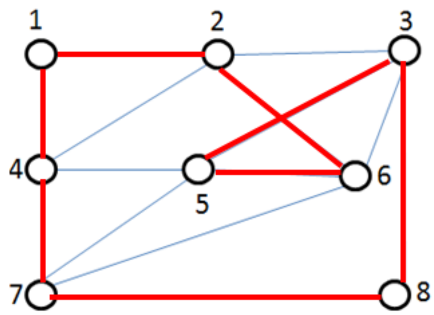
Solution: It has several Euler cycles, and since every Euler cycle is, by definition, an Euler path, it has an Euler path as well. An example of the sequence of edges that forms one of the possible Euler cycle is:

{8,7}, {7,4}, {4,1}, {1,2}, {2,3}, {3,6}, {6,7}, {7,5}, {5,4}, {4,2}, {2,6}, {6,5}, {5,3}, {3,8}

4. A Hamiltonian path is a path that uses every **node** of a graph exactly once. A Hamiltonian cycle is a Hamiltonian path that starts and ends at the same node (the only node that can be visited twice). Does the graph from the preceding question have a Hamiltonian path? Does it have a Hamiltonian cycle? If it does, then show it on the graph and if it does not then explain why.

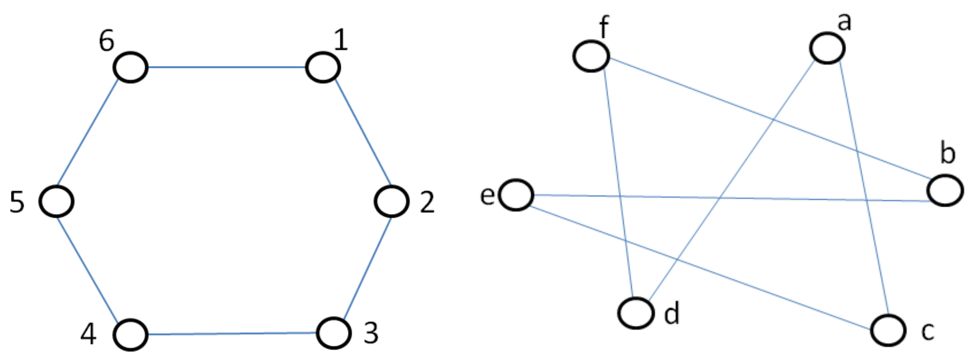
Solution: It has several Hamiltonian cycles as well, and since every Hamiltonian cycle is, by definition, a Hamiltonian path, it has a Hamiltonian path as well. If each edge is indicated using the vertex labels for the two vertices connected by that edge, the sequence of edges that forms one of the possible Hamiltonian cycles is: (graph is below)

{8,7}, {7,4}, {4,1}, {1,2}, {2,6}, {6,5}, {5,3}, {3,8}



Other possible solutions exist as well.

5. Are the following graphs isomorphic? Prove your answer by defining and comparing the vertices and edges of both graphs. If they are isomorphic, state which nodes and edges are equivalent.



The left graph is (V_L, E_L) where V_L = {1, 2, 3, 4, 5, 6} and E_L = {{1,2},{2,3},{3,4},{4,5},{5,6},{6,1}}
The right graph is (V_R, E_R) where V_R = {a, b, c, d, e, f} and E_R = {{a,c},{a,d},{b,e},{b,f},{c,e},{d,f}}.

If the vertices of the left graph were renamed such that:
Vertex "1" was renamed "a"
Vertex "2" was renamed "d"
Vertex "3" was renamed "f"
Vertex "4" was renamed "b"
Vertex "5" was renamed "e"
Vertex "6" was renamed "c"

Then the left graph would become: $V_{L'}$ = {a, c, d, e, b} and $E_{L'}$ = {{a,d},{d,f},{f,b},{b,e},{e,c}).
Since ($V_{L'}$, $E_{L'}$) = (V_R, E_R), the graphs must be isomorphic.

6. Determine whether or not the following are true and provide a full derivation explaining your answer. The domain of the functions of n below is the positive real

numbers. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.

Solutions

a. $(9n-5)^2$ is $\Theta(n^2)$ There are two acceptable solutions for Big O here.

Option 1:

$$if\ n \geq 1$$
$$81n^2 - 90n + 25 < 81n^2 + 90n + 25 \leq 81n^2 + 90n^2 + 25n^2 = 196n^2$$
$$\forall n \geq 1\ 81n^2 - 90n + 25 \leq 196n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 81n^2 - 90n + 25 \leq cn^2$$
$$f(x) \in O(g(x))$$

Option 2:

$$if\ n \geq 1$$
$$81n^2 - 90n + 25 < 81n^2 + 25 \leq 81n^2 + 25n^2 = 106n^2$$
$$\forall n \geq 1\ 81n^2 - 90n + 25 \leq 106n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 81n^2 - 90n + 25 \leq cn^2$$
$$f(x) \in O(g(x))$$

Omega:

$$if\ n \geq 1$$
$$81n^2 - 90n + 25 = 40n^2 + 41n^2 - 90n + 25$$
$$40n^2 + (41n^2 - 90n + 25) \geq 40n^2$$
$$\forall n \geq 1\ 81n^2 - 90n + 25 \geq 40n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 49n2 - 28n + 4 \geq cn^2$$
$$(x) \in \Omega(g(x))$$
$$(x) \in O(g(x)) \wedge f(x) \in \Omega(g(x))$$
$$(x) \in \Theta(g(x))$$

b. $2n^2 - 6 + n$ is $O(n^2)$

$$if\ n \geq 1$$
$$2n^2 + n - 6 < 2n^2 + n + 6 \leq 2n^2 + n^2 + 6n^2 = 9n^2$$
$$\forall n \geq 1\ 2n^2 + n - 6 \leq 9n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 2n^2 + n - 6 \leq cn^2$$
$$f(x) \in O(g(x))$$

c. $\frac{10\log(n+8)}{5}$ is $O(n^2)$

$$if\ n \geq 1$$
$$2\log(n+8) < 2(n+8) < 2n + 16 \leq 2n^2 + 16n^2 = 18n^2$$
$$\forall n \geq 1\ 2\log(n+8) \leq 18n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 2\log(n+8) \leq cn^2$$
$$f(x) \in O(g(x))$$

d. $1/n - 4$ is $O(n)$

$$if\ n \geq 1$$
$$1/n{-}4 < 1/n \leq n$$
$$\forall n \geq 1/n{-}4 \leq n$$
$$\exists c, k > 0\ \forall n \geq 1/n{-}4 \leq cn$$
$$f(x) \in O(g(x))$$

e. $6n^2$ is $\Omega(n^2)$

$$if\ n \geq 1$$
$$6n^2 \leq 6n^2$$
$$\forall n \geq 1\ 6n^2 \geq 6n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 6n^2 \geq cn^2$$
$$(x) \in \Omega(g(x))$$

f. $4n^2 - 3n$ is $\Omega(1)$

$$if\ n \geq 1$$
$$4n^2 - 3n \geq 4n^2 - 3n^2 = n^2$$
$$\forall n \geq 1\ 4n^2 - 3n \geq n^2$$
$$\exists c, k > 0\ \forall n \geq k\ 4n^2 - 6n \geq cn^2$$
$$(x) \in \Omega(g(x))$$

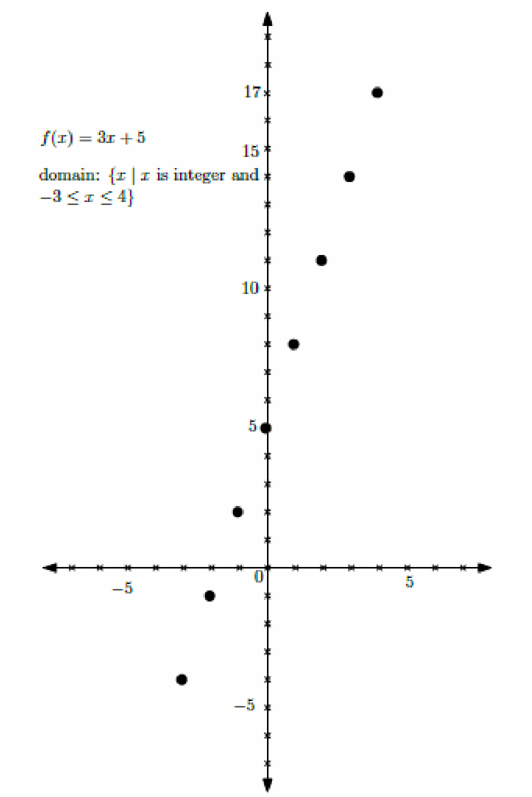
7. Suppose you know that an element is known to be among the first four elements in a list of 32 elements. Would a linear search or binary search locate this element more rapidly?
Solution: Linear
Since, worst case, for linear n=4, and worst case for binary, $\log_2(32)=5$

Also acceptable as a solution; IF you solved by cases, then the following is also ok:
If n=1,2,3 then Linear , if n=4 Binary

8. Describe how the number of comparisons used in the worst case changes when these algorithms are used to search for an element of a list when the size of the list doubles from n to $2n$, where n is a positive integer.
a) Linear search
Solution: doubles
b) Binary search
Solution: increases by 1

9. Describe how the number of comparisons used in the worst case changes when the size of the list to be sorted doubles from n to $2n$, where n is a positive integer when these sorting algorithms are used.
   a) Bubble sort
      Solution: $(2n)^2 = 4n^2$
   b) Selection Sort
      Solution: $(2n)^2 = 4n^2$

10. Plot the graph of f(x) = 3x + 5 where $\{x \mid x \in \mathbb{Z}, -3 \leq x \leq 4\}$.
    Solution: the solution must not be a line – it should just be a scatter plot. A line graph is incorrect.



11. Let a be the last digit in your student number, b be the second-to-last digit in your student number, $c = 5a$, and $d = 2b$. For the functions f and g defined as: $f: \mathbb{Z} \to \mathbb{Z}$; $f(x) = ax + c$ and $g: \mathbb{Z} \to \mathbb{Z}$; $g(x) = c^2 - d$, compute the following:
    Solutions: depends on your student ID (sub in the values below).
    a. $f \circ g$
       $f(g(x)) = f(c^2-d) = a(c^2-d) + c$
    b. $g \circ f$
       $g(f(x)) = c^2-d$
    c. $(f \circ g) \circ g$
       $f(g(g(x))) = f(g(c^2-d)) = f(c^2-d) = a(c^2-d) + c$