ENGR-233 SAMPLE EXAM, SOLUTIONS

Problem 1. Find the equation of the plane containing the points $P(2,1,0),\ Q(3,4,0),\ R(1,1,1)$.

Solution. $\overrightarrow{PQ}=\mathbf{i}+3\mathbf{j}$, $\overrightarrow{PR}=-\mathbf{i}+\mathbf{k}$; $\mathbf{a}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix}=3\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

Hence, the equation of the plane is $3(x-2)-(y-1)+3\mathrm{z}=0$, or $3x-y+3\mathrm{z}=5$.

(b) Find the parametric equation of the line through the point $S(3,4,0)$ which is perpendicular to the plane found in (a).

Solution. This line should be parallel to the vector $\mathbf{a}=3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ which is perpendicular to the plane found in (a). Hence its parametric equations are

$x=3+3\mathrm{t},\ y=4-t,\ z=3\mathrm{t}$.

Problem 2. (a) Let $w=\sqrt{x^3+y}+e^{xz}$, and let $x=2\mathrm{t},\ y=t^2,\ z-t^{-1}$. Use the multivariate chain rule to find $\frac{dw}{dt}$ for $t=1$.

Solution. $\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$

$$=\left[\frac{3x^2}{2\sqrt{x^3+1}}+ze^{xz}\right]\cdot 2+\left[\frac{1}{2\sqrt{x^3+y}}\right]+xe^{xz}\cdot\left(\frac{-1}{t^2}\right)$$

$$=\left[\frac{3\cdot 4t^2}{2\sqrt{8\mathrm{t}^3+t^2}}+t^{-1}e^2\right]+\left[\frac{1}{2\sqrt{8t^3+t^2}}\right]+2te^2\cdot\left(-\frac{1}{t^2}\right)\ ;$$

Upon substitution $t=1$ we obtain

$$\frac{dw}{dt}\Big|_{t=1}=\left(\frac{3\cdot 4}{2\sqrt{8+1}}+e^2\right)\cdot 2+\frac{1}{2\sqrt{8+1}}\cdot 2+2e^2\cdot(-1)=4+2e^2+\frac{1}{3}-2e^2=4\frac{1}{3}\ .$$

Problem 2 (b) Suppose the temperature distribution in the plane is given by

$T(x,y)=5\mathrm{x}^2+e^{xy}$ .If an ant is sitting at the point $P(2,3)$ in which unit direction should it move in order to cool off as soon as possible? What is the directional derivative of T in that direction?

Solution. $\operatorname{grad} T=(10x+ye^{xy})\boldsymbol{i}+xe^{xy}\boldsymbol{j}$. At the point $P(2,3)$, $\operatorname{grad} T=(20+3e^6)\boldsymbol{i}+2e^6\boldsymbol{j}$. The unit vector in the direction of the fastest cooling is

$$\boldsymbol{n}=-\frac{\operatorname{grad} T}{\|\operatorname{grad} T\|}=-\frac{(20+3e^6)\boldsymbol{i}+2e^6\boldsymbol{j}}{\sqrt{400+120e^6+13e^{12}}} .$$

The directional derivative in the direction $\boldsymbol{n}$ is

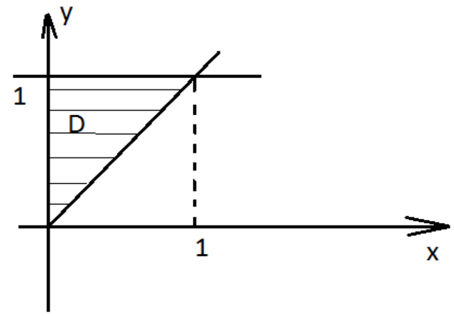
$$\frac{dT}{d\boldsymbol{n}}=-\|\operatorname{grad} T\|=-\sqrt{400+120e^6+13e^{12}} .$$

Problem 3. Find the point(s) on the surface $z=10-x^2-y^2$ at which the tangent plane is parallel to the plane $x+\frac{3}{2}y+\frac{1}{2}z+d=0$ where d is a constant.

Solution. The equation for the given plane can be written in the form $z=-2\mathrm{x}-3\mathrm{y}-2\mathrm{d}$. The equation of the plane tangent to the surface $z=f(x,y)$ at the point $(x_0,y_0,f(x_0,y_0))$ is $z=f(x_0,y_0)+\frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)$. In our case this equation is $z=10-x_0^2-y_0^2-2x_0(x-x_0)-2\mathrm{y}_0(y-y_0)=-2\mathrm{x}_0x-2\mathrm{y}_0y+(10+x_0^2+y_0^2)$. This plane is parallel to the plane $z=-2\mathrm{x}-3\mathrm{y}+d$ if $-2x_0=-2$, and $-2y_0=-3$. This means that $x_0=1,\ y_0=\frac{3}{2}$.

Problem 4. Find the moment of inertia about the y -axis of the lamina which is bounded by the lines $x=0,\ y=x,\ y=1$, and the density $\rho(x,y)=\sqrt{1+y^4}$.

Solution.



The inertia moment about the y -axis,

$$I_y=\iint_D x^2\rho(x,y)dA=\iint_D x^2\sqrt{1+y^4}dA .$$

Write this double integral as an iterated one, choosing x as an internal variable, and y as an external one (other order results in an intractable integral).

$I_y=\int_0^1\int_0^y x^2\sqrt{1+y^4}dx\,dy=\int_0^1\frac{y^3}{3}\sqrt{1+y^4}dy$. Let us substitute $t=y^4$, then $dt=4y^3dy$, $y^3dy=\frac{1}{4}dt$, and $I_y=\frac{1}{12}\int_0^1\sqrt{1+t}dt=\frac{1}{12}\cdot\frac{2}{3}(1+t)^{3/2}|_0^1=\frac{1}{18}(2^{3/2}-1)$.

Problem 5. For the scalar function $f(x,y,z)=e^{x^2}\cos z+z^4\sin z$ compute the following quantities **if they make sense**. If not, explain why.

(a) $\operatorname{grad} f=2\mathrm{x}e^{x^2}\cos z\boldsymbol{i}+z^4\cos y\boldsymbol{j}+(-e^{x^2}\sin z+4\mathrm{z}^3\sin y)\boldsymbol{k}$;

(b) $\operatorname{div}(\operatorname{grad} f)=(2+4\mathrm{x}^2)e^{x^2}-z^4\sin y+(-e^{x^2}\cos z+12\mathrm{z}^2\sin y)$;

(c) $\operatorname{grad}(\operatorname{div} f)$ is not defined because f is a scalar function, and therefore $\operatorname{div} f$ is not defined.

(d) $\operatorname{curl}(\operatorname{grad} f)=0$ (this is true for any regular function f).

(e) $\operatorname{grad}(\operatorname{grad} f)$ is not defined for $\operatorname{grad} f$ is a vector.

Problem 6. Compute the line integral $I=\int_C -ydx+xdy$ for the following curves:

(a) C is the curve from $(3,0)$ to $(-3,0)$ lying along the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ in the (x,y) -plane, $y\geq 0$.

Solution. The parametric equation of the half-ellipse is $x=3\cos t,\ y=2\sin t,\ 0\leq t\leq\pi$. Then $dx=-3\sin t\,dt,\ dy=2\cos t\,dt$, and

$$I=\int_0^\pi(-2\sin t\cdot-3\sin t+3\cos t\cdot 2\cos t)dt=\int_0^\pi 6(\cos^2 t+\sin^2 t)dt=6\pi\ .$$

(b) C is a closed curve in the (x,y) -plane whose equation in the polar coordinates is $r=2\cos\theta,\ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$.

Solution. The parametric equations of the curve C are

$x=2\cos^2\theta=1+\cos2\theta,\ \ y=2\cos\theta\sin\theta=\sin2\theta,\ \ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$. Then $dx=-2\sin2\theta d\theta,\ dy=2\cos2\theta d\theta$, and

$$I=\int_{-\pi/2}^{\pi/2}[(-\sin2\theta)\cdot(-2\sin2\theta)+(1+\cos2\theta)\cdot2\cos2\theta]d\theta$$

$$=\int_{-\pi/2}^{\pi/2}(2+2\cos2\theta)d\theta=2\pi+\sin2\theta|_{-\pi/2}^{\pi/2}=2\pi\ .$$

Problem 7. Prove that the line integral $I=\int_C(1+e^{-y})dx-(xe^{-y}+4\mathrm{y})dy$ is path independent, and the evaluate I from $(1,0)$ to $(2,1)$.

Solution. $\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=-e^{-y}+e^{-y}=0$. So, the integral is path independent, and $P=\frac{\partial f}{\partial x},\ \ Q=\frac{\partial f}{\partial y}$ for some function (named potential) f . Let us find the potential f .

$$\frac{\partial f}{\partial x}=1+e^{-y};\quad f=x(1+e^{-y})+g(y);$$

$$\frac{\partial f}{\partial y}=-xe^{-y}+g'(y)=Q=-xe^{-y}-4\mathrm{y};\ \ g'(y)=-4\mathrm{y};\ \ g(y)=-2\mathrm{y}^2+C.$$

So, $f(x,y)=x(1+4e^{-y})-2\mathrm{y}^2$. Now we find $f(1,0)=2;\ \ f(2,1)=2(1+e^{-1})-2=e^{-1}$.

So, $I=e^{-1}-2$.

Problem 8. Find the outward flux of the radial vector field $\boldsymbol{F}(x,y,z)-x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ through the boundary S of the domain D defined by the inequalities

$$x^2+y^2+z^2\leq2;\quad z\geq x^2+y^2\ .$$

Solution. By the Divergence Theorem, $\iint_S\boldsymbol{F}\cdot\boldsymbol{n}dS=\iiint_D\mathrm{div}\boldsymbol{F}dV$. For the given field $F,\ \ \mathrm{div}\boldsymbol{F}=3$, and so $\iint_S\boldsymbol{F}\cdot\boldsymbol{n}dS=2\iiint_D 3dV=3\,\mathrm{volume}(V)$. To find the volume of D , we find the intersection of the surfaces $x^2+y^2+z^2=2$ (sphere) and $z=x^2+y^2$

(paraboloid). Adding these two equations, we find $z^2+z-2=0;\quad z_1=1,\ z_2=-2$. We take the first root, and find the equations defining the curve: $x^2+y^2=1,\quad z=1$.

The volume of the domain D is defined by the following integral:

$V(D)=\iint_R(\sqrt{2-x^2-y^2}-(x^2+y^2))dA$ where R is the domain $R\colon x^2+y^2\le 1$. In the polar coordinates,

$$V(D)=\int_0^{2\pi}\int_0^1(\sqrt{2-r^2}-r^2)r\,dr\,d\theta=2\pi\cdot\frac{1}{2}\int_0^1(\sqrt{2-t}-t)dt=\pi\cdot[\frac{2}{3}(\sqrt{2}-1)-\frac{1}{2}]\ ;$$

Finally, $I=3V(D)=\pi\cdot[2(\sqrt{2}-1)-\frac{3}{2}]$.

Problem 9. Find the volume of the region $D\colon x^2+y^2+z^2\le 1,\ 4z^2\le x^2+y^2+z^{2,}\ z\ge 0$.

Solution. This region is a semi-ball with a cone deleted. So, it's natural to use the spherical coordinates. The conical part of the surface has the equation $3z^2=x^2+y^2$, or $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$; if α is the angle between the generator of this cone (one of the rays the cone is made of) and the z -axis, we find from the last equation that $\cos\alpha=\frac{1}{\sqrt{3}}$. Then, in the spherical coordinates (ϕ,θ,ρ) the volume is given by the integral

$$V(D)=\int_0^{2\pi}\int_\alpha^{\pi/2}\int_0^1\rho^2\sin\theta\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_\alpha^{\pi/2}\frac{1}{3}\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\frac{1}{3}\cos\alpha\,d\theta=\frac{2\pi}{3\sqrt{3}}\ .$$

Problem 10. (a) Find the work made by the force $\boldsymbol{F}=yz\boldsymbol{i}+xz\,\boldsymbol{j}+xy\boldsymbol{k}$ along the curve C defined by equations $x^2+y^2=1;\quad z=y^2$.

Solution. Using the Stokes theorem, we find $\int_C \boldsymbol{F}\,f\,\boldsymbol{r}=\iint_S \mathrm{curl}\boldsymbol{F}\cdot\boldsymbol{n}dS$ where S is any surface spanning the contour C . Now,

$\operatorname{curl}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ yz & xz & xy\end{vmatrix}=0\cdot\boldsymbol{i}+0\cdot\boldsymbol{j}+0\cdot\boldsymbol{k}=0.$ So, $\int\limits_{C}\boldsymbol{F}\cdot\boldsymbol{r}=0$ for any closed contour C .