Q1: A system is composed of four components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector (x1; x2; x3; x4) where xi is equal to 1 if component i is working and is equal to 0 if component i is failed.

a) How many outcomes are in the sample space of this experiment?

b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working. Specify all the outcomes in the event that the system works.

c) Let E be the event that components 1 and 3 are both failed. How many outcomes are contained in event E?

Solution:

(a) $2^4 = 16$

(b) $E = \{ (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1), (1,0,1,1), (0,1,1,1), (0,0,1,1) \}$

(c) Components 2 and 4 could be working or failed, so each of these components has two possible situations

$\#$ outcomes in $E = 2^2 = 4$

$E = \{ (0,0,0,0), (0,1,0,0), (0,0,0,1), (0,1,0,1) \}$

Q2: Each of 2 balls is painted black or gold and then placed in an urn. Suppose that each ball is colored black with probability 1/2, and that these events are independent.

a) Suppose that you obtain information that the gold paint has been used (and thus at least one of the balls is painted gold). Compute the conditional probability that both balls are painted gold.

b) Suppose, now, that the urn tips over and 1 ball falls out. It is painted gold. What is the probability that both balls are gold in this case?

Solution:

B = Ball is black, G = the ball is gold

$S = \{BB, BG, GB, GG\}$, each outcome has probability $\frac{1}{4}$

1G = at least one ball is gold $= \{BG, GB, GG\}$.

(a) $P(GG \mid 1G) = \frac{1}{3}$ since any coloring is equally likely

(b) FG = the balls that falls out is gold. By Bayesian formula

$$P(GG \mid FG) = \frac{P(FG \mid GG)\, P(GG)}{\cancel{P(FG)}\; P(FG \mid BB)\, P(BB) + P(FG \mid BG)\, P(BG) + P(FG \mid GB)\, P(GB) + P(FG \mid GG)\, P(GG)}$$

$$= \frac{1 \cdot \frac{1}{4}}{\frac{1}{4}\left(0 + \frac{1}{2} + \frac{1}{2} + 1\right)} = \frac{\frac{1}{4}}{\frac{1}{4} \times 2} = \frac{1}{2}.$$

Q3: While at Manchester United Football Club in England, Cristiano Ronaldo had about an 80% success rate of scoring penalty goals from his penalty kick attempts from the penalty spot. Assuming he kept scoring penalties at the same rate, and that all penalty kicks are independent, answer the following questions:

a) What is the probability that his first missed penalty kick will happen on his 4th penalty kick attempted?

b) What is the probability that Ronaldo would have to take more than 8 penalty kicks before missing his first attempt?

c) What is the probability that his second missed penalty kick will happen on his 8th penalty kick taken?

Solution:

(a) $P = (0.2)^3(0.8) =$

(b) $P = (0.8)^8$ (if we consider at least 8 kicks)

(c) $P = \binom{7}{1}(0.8)^6(0.2)^2$

Q4: For the random variable X with probability density function

$$f(x) = \begin{cases} c\,x^2(1-x) & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

a) Show that *c* must have the value *c = 12*
b) Draw an accurate graph of *f(x)*
c) What is the value of *E[X]*
d) Determine the distribution function *F(x)*
e) What is *P (X <= 1/2)*

Solution:

- Show that c must have the value $c = 12$.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 c \cdot x^2 (1-x)\,dx = c \cdot \int_0^1 (x^2 - x^3)\,dx$$

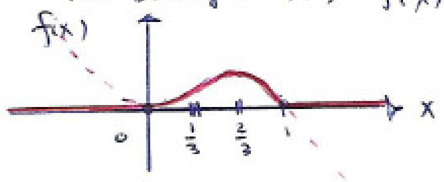
$$= c \cdot \left(\frac{x^3}{3} - \frac{x^4}{4} \right)\Big|_{x=0}^{x=1} = c \cdot \left(\frac{1}{3} - \frac{1}{4} \right) = c \cdot \frac{1}{12}$$

$$\Rightarrow \quad c = 12$$

- Draw an accurate graph of $f(x)$.

· $f(x) = 12 \cdot x^2 (1-x)$ has zeros at $x = 0$ & $x = 1$.

· For large x, $f(x) \cong -x^3 \Rightarrow \begin{cases} f \to -\infty \text{ as } x \to +\infty \\ f \to +\infty \text{ as } x \to -\infty \end{cases}$



· $f'(x) = 0 \Rightarrow 2x - 3x^2 = x(2-3x) = 0$

$\Rightarrow x = 0$ or $x = 2/3$

$\therefore$ loc. max at $x = 2/3$

- What is the value of $E[X]$?

$$\boxed{E[X] = 3/5}$$

$$E[X] = \int_{-\infty}^{+\infty} x\, f(x)\,dx$$

$$= 12 \cdot \int_0^1 x^3 (1-x)\,dx$$

$$= 12 \cdot \left(\frac{x^4}{4} - \frac{x^5}{5} \right)\Big|_{x=0}^{x=1} = 12 \cdot \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{12}{20}$$

- Determine the distribution function $F(x)$.

$\boxed{F(x) = x^3 (4-3x)}$ if $0 \le x \le 1$.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \begin{cases} 0 & x < 0 \\ \int_0^x 12 \cdot t^2 \cdot (1-t)\,dt = 12 \cdot \left(\frac{t^3}{3} - \frac{t^4}{4} \right)\Big|_{t=0}^{t=x} = 12 \cdot \left(\frac{x^3}{3} - \frac{x^4}{4} \right), & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

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i.e., $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3(4-3x) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \end{cases}$

- What is $P(X \le \frac{1}{2})$?

$$P(X \le \tfrac{1}{2}) = F(\tfrac{1}{2}) = (\tfrac{1}{2})^3 \cdot (4 - 3 \cdot \tfrac{1}{2})$$

$$\boxed{P(X \le \tfrac{1}{2}) = 5/16}$$

Q5: Suppose the probability that a randomly selected person is left-handed is 0.10. In a class of 250 students just how many left-handed seats should we have to be 95% sure that no left-handed person goes without a seat?

Solution:

Since we have a binomial random variable with parameter $p = 0.1$, $n = 250$

We define random variable X as the number of left handed students :

Approximating with a Gaussian density,

$$E(X) = np = 250 \times 0.1 = 25$$

$$var(X) = np(1-p) = 250 \times 0.1 \times 0.9 = 22.5$$

$$P(X < n_x) = P(X < n_x + 0.5) = 0.95$$

$$P(X < n_x + 0.5) = P\left(\frac{X - 25}{\sqrt{22.5}} < \frac{n_x + 0.5 - 25}{\sqrt{22.5}}\right)$$

$$= \Phi\left(\frac{n_x + 0.5 - 25}{\sqrt{22.5}}\right) = 0.95 \Rightarrow$$

$$\frac{n_x + 0.5 - 25}{4.74} = \Phi^{-1}(0.95) = 1.65 \Rightarrow$$

$$n_x = 1.65 \times 4.74 + 25 - 0.5 = 32.32 \Rightarrow \boxed{n_x = 33}$$

Q6: A six-sided die, in which each side is equally likely to appear, is repeatedly rolled until the total of all rolls exceeds 400. Approximate the probability that this will require more than 140 rolls.

Solution:

rolling a die once, is a random variable with the following mean and variance

$P(x_i) = \frac{1}{6} \quad i = 1, \ldots, 6$

$\mu_x = \sum_{i=1}^{6} i \times P_i = \frac{1+2+\cdots+6}{6} = 3.5$

$\sigma_x^2 = \sum_{i=1}^{6} (x_i - \mu)^2 P_i =$

$\frac{(1-3.5)^2 + (2-3.5)^2 + \cdots + (6-3.5)^2}{6} = 2.917$

The probability that this will require more than 140 rolls is equall to the probability that the total of 140 rolls is less than 400:

$y = x_1 + \cdots + x_{140}$

$P(y \leqslant 400) = P\left(\frac{y - \mu_y}{\sigma_y} \leqslant \frac{400 - \mu_y}{\sigma_y}\right)$

$\mu_y = 140\,\mu_x = 140 \times 3.5 = 490$

$\sigma_y = \sqrt{140 \times 2.917} = 20.21$

$= P\left(\frac{y - 490}{20.21} \leqslant \frac{400 - 490}{20.21}\right) = \Phi(-4.45) =$

$1 - \underbrace{\Phi(4.45)}_{\simeq 1} \simeq 0$

Q7: The lifetime of a certain type of emergency batteries is normally distributed. A sample of 64 batteries has produced a sample mean of 756.4 hours and a sample standard deviation of 33.9 hours.

(a) Test the null hypothesis H_0: $\mu \leq 747.5$ versus alternate hypothesis H_1: $\mu > 747.5$ at the significance level of $\alpha_1 = 0.05$.

(b) Test the null hypothesis H_0: $\mu \leq 747.5$ versus alternate hypothesis H_1: $\mu > 747.5$ at the significance level of $\alpha_1 = 0.01$.

(c) Find the p-value of the test, and use the p-value to verify your answers

Solution:

(a) We notice that $n = 64$, so we can assume by the Central Limit Theorem that $\sigma \approx S = 33.9$. If the null hypothesis is true, then the distribution of

$$\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{X} - 747.5}{33.9/\sqrt{64}} = Z,$$

is approximately standard normal. We notice that

$$z_{0.05} = 1.645 < z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{756.4 - 747.5}{33.9/8} \approx 2.10.$$

Hence, H_0 is rejected at $\alpha = 0.05$ (since $2.10 > 1.645$).

(b) We also notice that

$$z_0 \approx 2.10 < z_{0.01} = 2.327.$$

Hence, H_0 is accepted at $\alpha = 0.01$ (since $2.327 > 2.10$).

(c) The p-value of the test is

$$P\{Z > z_0\} = P\{Z > 2.10\} = 1 - P\{Z \leq 2.1\} \approx 1 - 0.9821 \approx 0.018 = 1.8\%.$$

Hence, the p-value is smaller than 0.05 (reject) and greater than 0.01 (accept).

Q8: Suppose $X_1, X_2 \ldots X_n$, are identical, independent, normal random variables, with mean = 7 and standard deviation = 4. Let

$$\bar{X} \equiv \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

Use the CLT to determine at least how big n must be so that

$$P(|\ \bar{X} - \mu\ | \ \leq\ 1)\ \geq\ 90\ \%\ .$$

Solution

$X_1, X_2, \cdots, X_n$, are identical, independent, *normal* , on $(-\infty, \infty)$.

Each X_i has mean $\mu = 7$, and standard deviation $\sigma = 4$.

Thus $\bar{X}$ has mean $\mu_{\bar{X}} = 7$, and standard deviation $\sigma_{\bar{X}} = \frac{4}{\sqrt{n}}$.

We have

$$P(|\ \bar{X} - \mu\ |\ \leq\ 1)\ =\ 1\ -\ 2\ \Phi(\frac{6-7}{4/\sqrt{n}})\ =\ 1\ -\ 2\ \Phi(-\frac{\sqrt{n}}{4})\ ,$$

so we need n so that

$$\Phi(-\frac{\sqrt{n}}{4})\ \leq\ 0.05\ .$$

From the Standard Normal Table we find that is sufficient to take

$$\frac{\sqrt{n}}{4}\ =\ 1.65\ , \quad \text{which gives} \quad n\ =\ 44\ .$$