

## ENGR 213 Final Exam (Fall 2013)

### **Problem 1:**

Given that  $y = -\frac{2}{x} + x$ , is a solution of the differential equation  $xy' + y = 2x$ . Find  $x_0$  and the largest interval  $I$  for which  $y(x)$  is a solution of the initial-value problem:

$$xy' + y = 2x, \quad y(x_0) = 1$$

### **Problem 2:**

Show that the given differential equation is exact and obtain the solution:

$$(x^3 + y^3)dx + 3xy^2dy = 0$$

### **Problem 3:**

Find all cube roots of  $z = 27i$ .

### **Problem 4:**

Solve the following initial value problem

$$y'' + 6y' + 8y = \sin(2x), \quad y(0) = 0, \quad y'(0) = 0$$

by using undetermined coefficients.

### **Problem 5:**

Find general solution by variation of parameters:

$$x^2 y'' - xy' + y = x$$

### **Problem 6:**

Find the first eight coefficients (i.e.  $a_0, a_1, a_2, \dots, a_7$ ) of the power series expansion

$$y = \sum_{n=0}^{\infty} a_n x^n$$

of the solution to the differential equation

$$y'' + xy' + y = 0$$

subject to the initial value conditions  $y(0) = 0$ ,  $y'(0) = 1$

**Problem 7:**

Write in matrix form and find the general solution of the system:

$$\frac{dx}{dt} = 6x - y$$

$$\frac{dy}{dt} = 5x + 2y$$

Final Examination

1.  $y = -\frac{2}{x} + x$  is solution of  $xy' + y = 2x$

$$y(x_0) = 1.$$

a)  $1 = -\frac{2}{x_0} + x_0$  i.e.  $x_0^2 - x_0 - 2 = 0$

$$x_0 = 2 \text{ or } -1.$$

b) Larger interval is

$$\underline{\underline{-1 < x_0 < \infty}}$$

2.  $(x^3 + y^3) dx + 3xy^2 dy = 0.$

$$M dx + N dy = 0.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow 3y^2 = 3y^2 \therefore \text{Exact DE.}$$

$$f = \int M dx + g(y)$$

$$= \frac{x^4}{4} + xy^3 + g(y)$$

$$\frac{\partial f}{\partial y} = 3xy^2 + g'(y) = N = 3xy^2$$

$$\therefore g'(y) = 0, \quad g(y) = \text{constant.}$$

$$\underline{\underline{f = \frac{x^4}{4} + xy^3 + C}}$$

3. Cube root of  $27i$

$$27i = 27 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$\therefore \sqrt[3]{27i} = 3 \left[ \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right],$$

$$3 \left[ \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right],$$

$$3 \left[ \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} \right].$$

4.  $y'' + 6y' + 8y = \sin 2x$        $y(0) = 0, y'(0) = 0.$

$$y = e^{mx}$$

$$m^2 + 6m + 8 = 0 \quad \text{Homogeneous part.}$$

$$(m+2)(m+4) = 0 \quad m_1 = -2, m_2 = -4$$

$$y_c = C_1 e^{-2x} + C_2 e^{-4x}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y_p' = -2A \sin 2x + 2B \cos 2x$$

$$y_p'' = -4A \cos 2x - 4B \sin 2x$$

Substituting

$$-4A \cos 2x - 4B \sin 2x - 12A \sin 2x$$

$$+ 12B \cos 2x + 8A \cos 2x + 8B \sin 2x = \sin 2x$$

$$\therefore -4B - 12A + 8B = 1$$

$$-4A + 12B + 8A = 0$$

Solving  $B = \frac{1}{40}$ ,  $A = -\frac{3}{40}$ . (3)

$$\therefore y = c_1 e^{-2x} + c_2 e^{-4x} - \frac{3}{40} \cos 2x + \frac{1}{40} \sin 2x.$$

Substituting initial conditions and solving for  $c_1$  &  $c_2$

$$c_1 = \frac{1}{8}, \quad c_2 = -\frac{1}{20}.$$

$$y = \frac{1}{8} e^{-2x} - \frac{1}{20} e^{-4x} - \frac{3}{40} \cos 2x + \frac{1}{40} \sin 2x.$$

5.

$$x^2 y'' - xy' + y = x$$

$y = x^m$ . Homogeneous case  $\rightarrow$

$$\therefore m^2 - 2m + 1 = 0 = (m-1)^2$$

$$\therefore y_c = c_1 x + c_2 x \ln x \quad \text{i.e. } c_1 y_1 + c_2 y_2$$

Expressing in standard form

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{1}{x} \underbrace{\quad}_{f(x)}$$

$$W = \begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$$

$$u_1' = \frac{-y_2 f(x)}{W} = -\frac{\ln x}{x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{1}{x}$$

(4)

Integrate:

$$u_1 = - \int \frac{\ln x}{x} dx = - \frac{\ln x^2}{2} + C$$

$$u_2 = \int \frac{1}{x} dx = \ln x + C$$

$$y_p = u_1 y_1 + u_2 y_2 = - \frac{\ln x^2}{2} \cdot x + \ln x (x \ln x)$$

$$\therefore y = C_1 x + C_2 x \ln x - x \cdot \frac{\ln x^2}{2} + x (\ln x)^2.$$


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6.  $y'' + xy' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$\therefore \sum_{n=2}^{\infty} a_n n \cdot (n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n \cdot n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{k=0}^{\infty} a_{k+2} \cdot (k+2)(k+1) x^k + \sum_{k=1}^{\infty} a_k \cdot k \cdot x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\therefore 2a_2 + a_0 + \sum_{k=1}^{\infty} [a_{k+2} (k+2)(k+1) + (k+1)a_k] x^k = 0.$$

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$$2a_2 + a_0 = 0$$

$$a_{k+2} = \frac{-(k+1)a_k}{(k+2)(k+1)} = -\frac{a_k}{k+2}$$

$$\therefore a_2 = -\frac{a_0}{2} \quad \text{--- (1)}$$

$$a_3 = \frac{-a_1}{3} \quad \text{(2)}$$

$$a_4 = \frac{-a_2}{4} \quad \text{--- (3)}$$

$$\therefore y = a_0 \left[ 1 - \frac{1}{2}x^2 + \dots \right] + a_1 \left[ x - \frac{1}{3}x^3 + \dots \right]$$

$$y(0) = a_0 = 1$$

$$y'(0) = a_1 = 1$$

$$\therefore y_0 = \left[ 1 - \frac{x^2}{2} + \dots \right] + \left[ x - \frac{x^3}{3} + \dots \right]$$

7.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{Matrix form.}$$

$$\begin{vmatrix} (6-\lambda) & -1 \\ 5 & (2-\lambda) \end{vmatrix} = 0 \quad \lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$$

$$(6-\lambda)x - y = 0$$

$$\therefore (6-\lambda)x = y$$

$$\lambda = 4 - i \rightarrow (6 - 4 + i)x = y$$

$$\underline{\underline{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}}}$$

$$\lambda = 4 + i \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$