COMP 232 Tutorial 6

March 10, 2017

Exercise 2.3 Question 30

Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(s)$ if

a) $f(x) = 1$
Ans: $f(S) = \{1\}$

b) $f(x) = 2x + 1$
Ans: $f(S) = \{-1, 1, 5, 9, 15\}$

c) $f(x) = \lceil x/5 \rceil$
Ans: $f(S) = \{0, 1, 2\}$

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$
Ans: $f(S) = \{0, 1, 5, 16\}$

Question 32

Let $f(x) = 2x$ where the domain is the set of real numbers. What is

a) $f(\mathbb{Z})$
Ans: Even numbers

b) $f(\mathbb{N})$
Ans: Positive even numbers

c) $f(\mathbb{R})$
Ans: $\mathbb{R}$

Question 40

Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$
Ans: Equivalent to $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$
Prove $f(S \cup T) \subseteq f(S) \cup f(T)$

$$
\begin{aligned}
&\forall y \exists x (y \in f(S \cup T) \rightarrow x \in S \cup T \wedge f(x) = y) \\
\Rightarrow &\forall y \exists x (y \in f(S \cup T) \rightarrow (x \in S \vee x \in T) \wedge f(x) = y) \\
\Rightarrow &\forall y \exists x (y \in f(S \cup T) \rightarrow (x \in S \wedge f(x) = y) \vee (x \in T \wedge f(x) = y) \\
\Rightarrow &\forall y (y \in f(S \cup T) \rightarrow (y \in f(S) \vee y \in f(T))) \\
\Rightarrow &\forall y (y \in f(S \cup T) \rightarrow (f(S) \vee f(T))) \\
\Rightarrow &\forall f(S \cup T) \subseteq (f(S) \vee f(T))
\end{aligned}
$$

Prove $f(S) \cup f(T) \subseteq f(S \cup T)$

$$
\begin{aligned}
&(S \subseteq S \cup T) \wedge (T \subseteq S \cup T) \\
\Rightarrow &(f(S) \subseteq f(S \cup T)) \wedge (f(T) \subseteq f(S \cup T)) \\
\Rightarrow &f(S) \cup f(T) \subseteq f(S \cup T)
\end{aligned}
$$

a) $f(S \cap T) \subseteq f(S) \cap f(T)$
Ans:

$$
\begin{aligned}
&(S \cap T \subseteq S) \wedge (S \cap T \subseteq T) \\
\Rightarrow &(f(S \cap T) \subseteq f(S)) \wedge (f(S \cap T) \subseteq f(T)) \\
\Rightarrow &f(S \cap T) \subseteq f(S) \cap f(T))
\end{aligned}
$$

Question 70

Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.
Ans:

$$
\begin{aligned}
&\forall x \in X (x \in (f \circ g)^{-1}(Z) \leftrightarrow \exists z \in Z (f \circ g(x) = z)) \\
\Leftrightarrow &\forall x \in X (x \in (f \circ g)^{-1}(Z) \leftrightarrow \exists z \in Z, y \in Y (f(y) = z \wedge g(x) = y)) \\
\Leftrightarrow &\forall x \in X (x \in (f \circ g)^{-1}(Z) \leftrightarrow \exists z \in Z, y \in Y (f^{-1}(z) = y \wedge g(y) = x)) \\
\Leftrightarrow &\forall x \in X (x \in (f \circ g)^{-1}(Z) \leftrightarrow \exists z \in Z (f^{-1} \circ g(y) = x))
\end{aligned}
$$

Question 71

Let S be a subset of a universal set U. The **characteristic function** f_s of S is the function from U to set $\{0, 1\}$ such that $f_s(x) = 1$ if x belongs to S and $f_s(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all $x \in U$

a) $f_{A\cap B}(x) = f_A(x) \cdot f_B(x)$
Ans:

$$\begin{aligned} & f_{A\cap B}(x) = 1 \\ \Leftrightarrow & x \in A \cap B \\ \Leftrightarrow & x \in A \wedge x \in B \\ \Leftrightarrow & f_A(x) = 1 \wedge f_B(x) = 1 \\ \Leftrightarrow & f_A(x) \cdot f_B(x) = 1 \end{aligned}$$

b) $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
Ans:

$$\begin{aligned} & f_{A\cup B}(x) = 1 \Leftrightarrow && x \in A \cup B \\ \Leftrightarrow & x \in A \vee x \in B \\ \Leftrightarrow & f_A(x) = 1 \vee f_B(x) = 1 \\ \Leftrightarrow & f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 1 \end{aligned}$$

c) $f_{\overline{A}}(x) = 1 - f_A(x)$
Ans:

$$\begin{aligned} & f_{\overline{A}}(x) = 1 \\ \Leftrightarrow & x \in \overline{A} \\ \Leftrightarrow & x \notin A \\ \Leftrightarrow & f_A x = 0 \\ \Leftrightarrow & 1 - f_A x \end{aligned}$$

d) $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x)$
Ans:

$$\begin{aligned} & f_{A\oplus B}(x) = 1 \\ \Leftrightarrow & x \in A \oplus B \\ \Leftrightarrow & (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B) \\ \Leftrightarrow & (f_A(x) = 1 \wedge f_B(x) = 0) \vee (f_A(x) = 0 \wedge f_B(x) = 1) \\ \Leftrightarrow & (f_A(x) + f_B(x) = 1) \wedge (f_A(x) f_B(x) = 0) \\ \Leftrightarrow & f_A(x) + f_B(x) - 2f_A(x) f_B(x) = 1 \end{aligned}$$

Question 72

Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

Ans: one-to-one $\Rightarrow$ onto
Since f is one-to-one, $|f(A)| = |A|$. Then $|f(A)| = A = B$. Thus, f is onto.
onto $\Rightarrow$ one-to-one
Since f is onto, $|f(A)| = |B|$. Thus, $|f(A)| = B = A$. Thus, f is one-to-one.

Question 73

Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$
Ans: True. $\lceil x \rceil$ is an integer. So, $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$

b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ whenever x is a real number.
Ans: False. Let $x = 0.6$

c) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ or 1 whenever x and y are real numbers.
Ans: True. Let $x = x_1 + x_2$ and $y = y_1 + y_2$, $x_1, y_1 \in \mathbb{Z}$ and $0 \leq x_2, y_2 < 1$.
If $x_2, y_2 = 0$, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x_1 + y_1 - x_1 - y_1 = 0$
If $x_2 = 0, y_2 \neq 0$ or $x_2 \neq 0, y_2 = 0$, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x_1 + y_1 + 1 - x_1 - y_1 - 1 = 0$
If $x_2 \neq 0, y_2 \neq 0, x_2 + y_2 \leq 1$, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x_1 + 1 + y_1 + 1 - x_1 - y_1 - 1 = 1$
If $x_2 \neq 0, y_2 \neq 0, 1 < x_2 + y_2 < 2$, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x_1 + 1 + y_1 + 1 - x_1 - y_1 - 2 = 0$

d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all real numbers x and y.
Ans: False. Let $x = 2$ $y = 0.1$

e) $\lceil \frac{x}{2} \rceil = \lfloor \frac{x+1}{2} \rfloor$ for all real numbers x.
Ans: False. Let $x = 0.1$

Question 77

For each of these partial functions, determine its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether it is a total function.

a) $f : \mathbb{Z} \to \mathbb{R}, f(n) = 1/n$
Ans:
Domain: $\mathbb{Z}$
Codomain: $\mathbb{R}$
Domain of definition: Nonzero integers
Set of undefined values: $\{0\}$
Total function: No

b) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = \lceil n/2 \rceil$ Ans:
Domain: $\mathbb{Z}$
Codomain: $\mathbb{Z}$
Domain of definition: $\mathbb{Z}$
Set of undefined values: $\emptyset$
Total function: Yes

c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}, f(m, n) = m/n$ Ans:
Domain: $\mathbb{Z} \times \mathbb{Z}$
Codomain: $\mathbb{Q}$
Domain of definition: $\mathbb{Z} \times (\mathbb{Z} - \{0\})$
Set of undefined values: $\mathbb{Z} \times \{0\}$
Total function: No

d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, f(m, n) = mn$ Ans:
Domain: $\mathbb{Z} \times \mathbb{Z}$
Codomain: $\mathbb{Z}$
Domain of definition: $\mathbb{Z} \times \mathbb{Z}$
Set of undefined values: $\emptyset$
Total function: Yes

e) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, f(m, n) = m - n$ if $m > n$ Ans:
Domain: $\mathbb{Z} \times \mathbb{Z}$
Codomain: $\mathbb{Z}$
Domain of definition: $\{(m, n) | m > n\}$
Set of undefined values: $\{(m, n) | m \leq n\}$
Total function: No