

# MAT 2355 Fall 2013 Final Exam

11-December, 2013.      Duration: 3 hours

Instructor: Barry Jessup

Family Name: \_\_\_\_\_

First Name: \_\_\_\_\_

Student number: \_\_\_\_\_

1	
2	
3	
4	
5	
6	
7 (Bonus)	
Total	

**PLEASE READ THESE INSTRUCTIONS CAREFULLY BEFORE BEGINNING:**

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. Calculators are allowed, but may only be used for computations and not for storage of data or text. The use of communication devices, or any other image or text storage device is not permitted.
3. The correct answer requires reasonable justification written legibly and logically –you must convince me that you know why your solution is correct.
4. Questions 1–6 are each worth an equal number of points. Question 7 is a bonus question worth a maximum of 5% extra and should not be tried until all parts of questions 1-6 have been completed and checked.
5. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
6. Good luck! Bonne chance!

1. Consider the following 3 lines in the plane:

$$L_1 = \{(x, y) \in \mathbf{R}^2 \mid x = 0\}$$

$$L_2 = \{(x, y) \in \mathbf{R}^2 \mid x + y = -1\}$$

$$L_3 = \{(x, y) \in \mathbf{R}^2 \mid x - y = 0\}$$

- a) Carefully state the “Three reflections Theorem” for the Euclidean plane.
- b) Sketch  $L_1, L_2$  and  $L_3$  carefully on the same set of labelled axes.
- c) Completely describe the isometry  $f = R_{L_3}R_{L_2}R_{L_1}$ , and justify your answer. (You do not need to compute a formula for  $f$ .)
- d) Using (b), or otherwise, completely describe the isometry  $g = f \circ f$ . Justify your answer. (You do not need to compute a formula for  $g$ .)

2. a) If  $X$  and  $Y$  are subsets of  $\mathbf{R}^3$ , define what is meant by ' $X$  is congruent to  $Y$ .'
- b) If  $A, B \in \mathbf{R}^3$ , define the segment  $\overline{AB}$ .
- c) Prove that if  $f$  is an isometry of  $\mathbf{R}^3$ , with  $f(A) = E$  and  $f(B) = F$ , then  $f(\overline{AB}) = \overline{EF}$ .
- d) Using only reflections, show that any two segments  $\overline{AB}$  and  $\overline{EF}$  with  $\|A - B\| = \|E - F\|$  are congruent.
- e) Suppose two triangles  $\Delta ABC$  and  $\Delta EFG$  in  $\mathbf{R}^n$  satisfy

$$\|A - B\| = \|E - F\|, \quad \|B - C\| = \|F - G\|, \quad \text{and} \quad \|A - C\| = \|E - G\|.$$

Prove carefully that the two triangles  $\Delta ABC$  and  $\Delta EFG$  are congruent. Note that you are being asked to prove 'SSS' in  $\mathbf{R}^3$  (*Hint: Use part (d) first.* .)

3. Let  $C_1 = \{v \in \mathbf{S}^2 \mid (-1, 1, 0) \cdot v = 0\}$  and  $C_2 = \{v \in \mathbf{S}^2 \mid (1, 0, 0) \cdot v = 0\}$  be great circles on  $\mathbf{S}^2$ .

a) Find  $C_1 \cap C_2$ .

b) Find formulae for the reflections  $R_{C_1}$  and  $R_{C_2}$  in  $C_1$  and  $C_2$  respectively.

c) Use your answers from (b) to find  $A \in \mathbf{O}(3)$  so that  $R_{C_2}R_{C_1}(v) = Av$ .

d) Find a point  $a \in \mathbf{S}^2$  and an angle  $\theta \in [0, 2\pi)$  such that  $R_{C_2}R_{C_1}$  is the rotation by  $\theta$  in the positive sense about the point  $a \in \mathbf{S}^2$ . Use the formula  $R_{a,\theta}(v) = (v \cdot a)(1 - \cos \theta)a + (\cos \theta)v + (\sin \theta)(a \times v)$  to check your answer.

4. Let  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ .

- a) Show that  $g$  defined by  $g(v) = Av$  is an isometry of  $\mathbf{S}^2$ .
- b) Determine the isometry type of  $g$ , giving all details.
- c) Express  $g$  as a product of reflections in great circles.
- d) Find a vector  $v_0 \in \mathbf{R}^3$  such that  $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  defined by  $f(v) = g(v) + v_0$  is not a reflection. (You must justify your answer.)

5. The latitude and longitude of Brussels, Belgium and Auckland, New Zealand are given below:

	Latitude	Longitude
Brussels	$50^{\circ}52' N$	$4^{\circ}22' E$
Auckland	$36^{\circ}52' S$	$174^{\circ}45' E$

- a) Find the  $(\theta, \phi)$  coordinates of Brussels and Auckland from the given data. Express your answer in radians, correct to 4 significant figures.
- b) Let  $N$ ,  $A$  and  $B$  respectively denote the North pole, Auckland and Brussels and consider the spherical triangle  $\triangle NAB$ . Find the vertex angle  $\hat{N}$  in degrees and minutes, correct to the nearest minute. (Draw yourself a picture looking down from the north pole and remember that  $0 \leq \hat{N} \leq 180^{\circ}$ .)
- c) Find the shortest distance from Brussels to Auckland. Assume that the earth is a sphere of radius 6380 km. Express your answer in kilometres, correct to 3 significant figures.
- d) If you were to fly from Auckland to Brussels along the arc of the great circle  $\overline{AB}$ , find your heading as you leave Auckland. Express the angle part of your answer in degrees and minutes, correct to the nearest minute.
- e) If you were to start at Auckland, maintain the heading found in (d), and travel at a constant speed of 1000 kilometres per hour, where would you end up, and how long would it take you to get there? Express your answer to the second part in hours and minutes, correct to the nearest minute.

6. a) Find a formula for the stereographic projection

$$f : \mathbf{S}^2 \setminus \{(0, 0, 1)\} \rightarrow H = \{(x, y, 0) \mid x, y \in \mathbf{R}\}.$$

- b) Let  $C$  be any great circle on  $\mathbf{S}^2$  containing  $(0, 0, 1)$ . Describe the image of  $C \setminus \{(0, 0, 1)\}$  under the stereographic projection, giving all details.
- c) Give an example  $C$  of a great circle on  $\mathbf{S}^2$  not containing  $(0, 0, 1)$  whose image under the stereographic projection is not a straight line.
- d) Which (if any) of the following are true for  $f$ ? Give some justification. (You may use results from class.)
- i) it is an isometry
  - ii) it preserves area
  - iii) it preserves angles
  - iv) it preserves the shape of geodesics

7. (Bonus) Suppose  $\rho$  and  $\rho'$  are rotations of  $\mathbf{S}^2$  such that  $\rho\rho' = \rho'\rho$ . Show that either
- i)  $\rho$  and  $\rho'$  are rotations about the same points, or,
  - ii)  $\rho$  and  $\rho'$  are rotations, both by  $\pi$ , about perpendicular points.