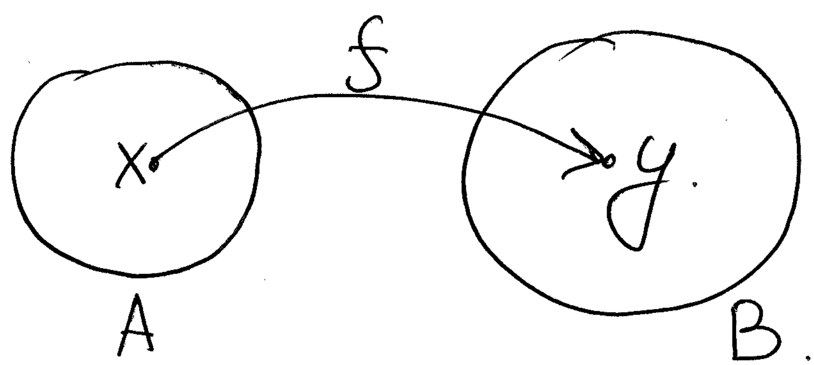
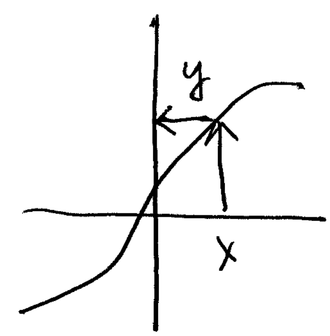
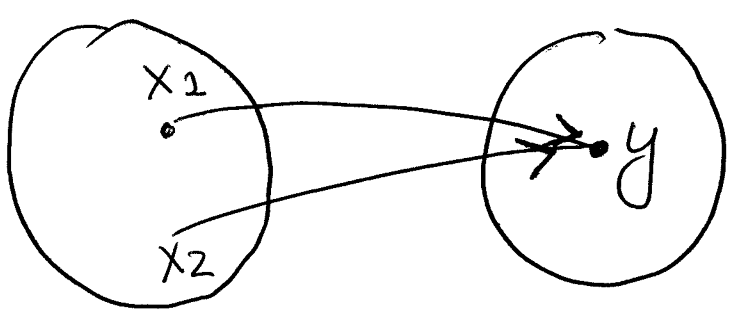
Functions (Review, Part 1).

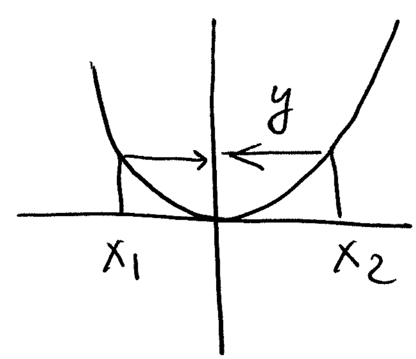
$\boxed{\text{Definition}}$ A function f is a rule that assigns each element x in the set A (the domain of f) $\boxed{\text{exactly one}}$ element y in the set B (the range of f)

$$f : A \longrightarrow B$$









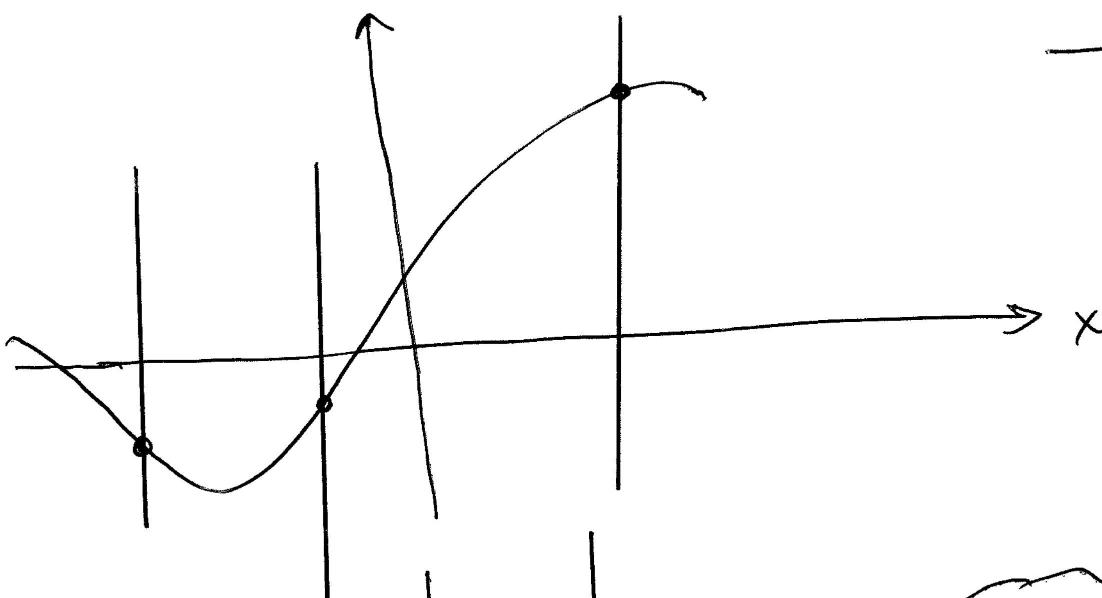
x is the independent variable

y is the dependent variable.

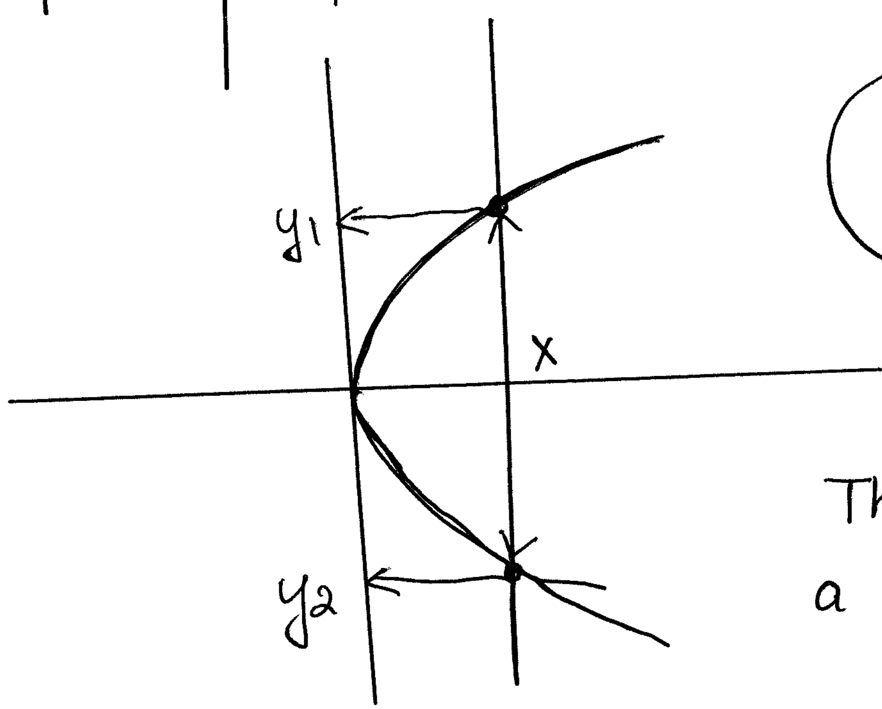
We graph $y=f(x)$ in the xy plane.

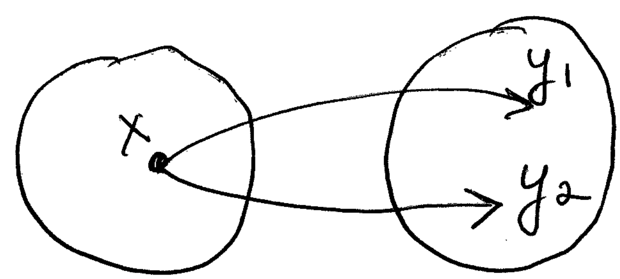
There is a simple test to decide whether or not $f(x)$ is a <u>function</u>.

The Vertical Line Test



— a function

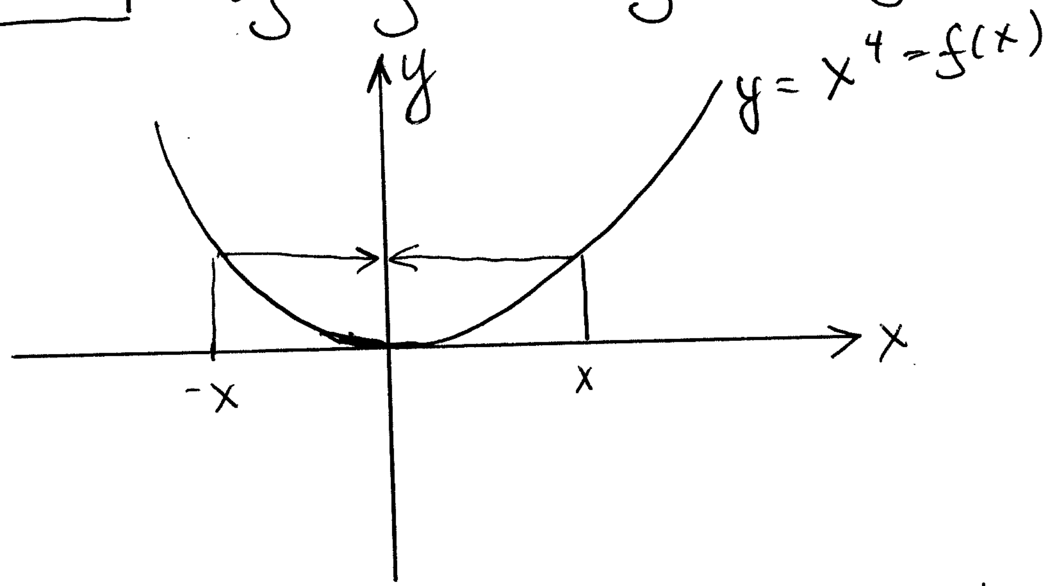




The relation is not a function!

A function f, $f: A \to B$ is called (3)

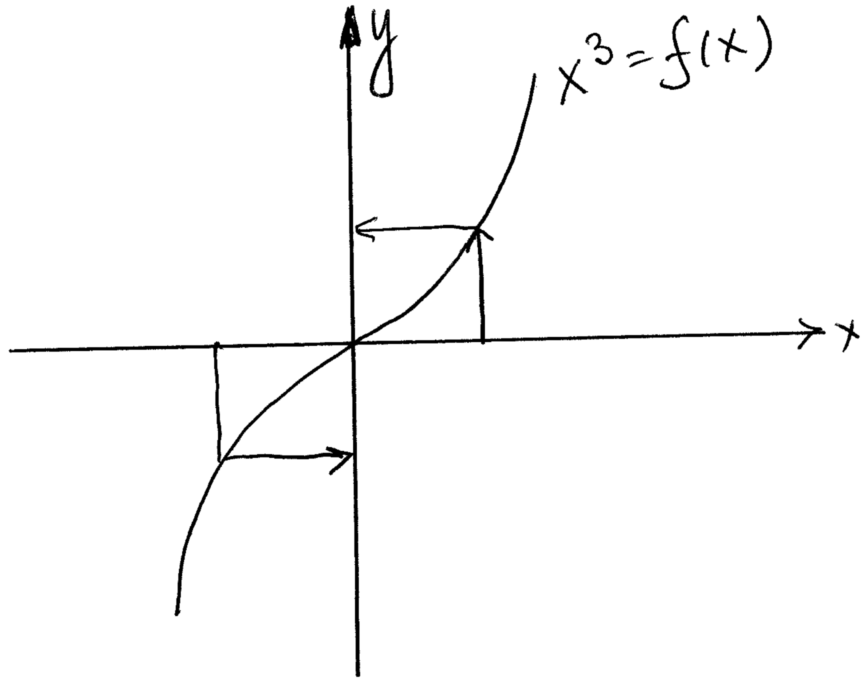
(1) $\boxed{\text{even}}$ if $f(x) = f(-x)$ for all $x \in A$



(symmetric about the y-axis)

---

(2) $\boxed{\text{odd}}$ if $f(x) = -f(-x)$ for all $x \in A$.



(symmetric about the origin)

FSM 1 Problem 16.

Can a function be even and odd at the same time?
If yes, give an example

$$-f(-x) \overset{\text{odd}}{=} f(x) \overset{\text{even}}{=} f(-x) \implies$$

$$-f(-x) = f(-x) \quad \text{or}$$

$$2f(-x) = 0$$

$$f(-x) = 0$$

We know, $f(-x) = f(x)$.

Thus, $\boxed{f(x) \equiv 0}$

Common functions

(1) A polynomial function is a function of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

→ $a_0, a_1 \dots a_n$ are real-valued constants

→ n is nonnegative integer (the degree of the polynomial)

$$f(x) = 4x^3 + 3x - 5$$

$A = \mathbb{R}$ (the domain is all real numbers).

2 A rational function

$$f(x) = \frac{P(x)}{q(x)}, \quad \text{where}$$

$P(x)$, $q(x)$ are two polynomia functions

$\boxed{q(x) \neq 0.}$

<u>Example</u>: $f(x) = \dfrac{x^2+x-2}{x^2-x-2} = \dfrac{P(x)}{q(x)}$

$q(x) = x^2 - x - 2 = 0$

$x^2 - x - 2 = (x+1)(x-2) = 0.$

$A = \{x \in \mathbb{R} : x \neq -1, x \neq 2\}.$

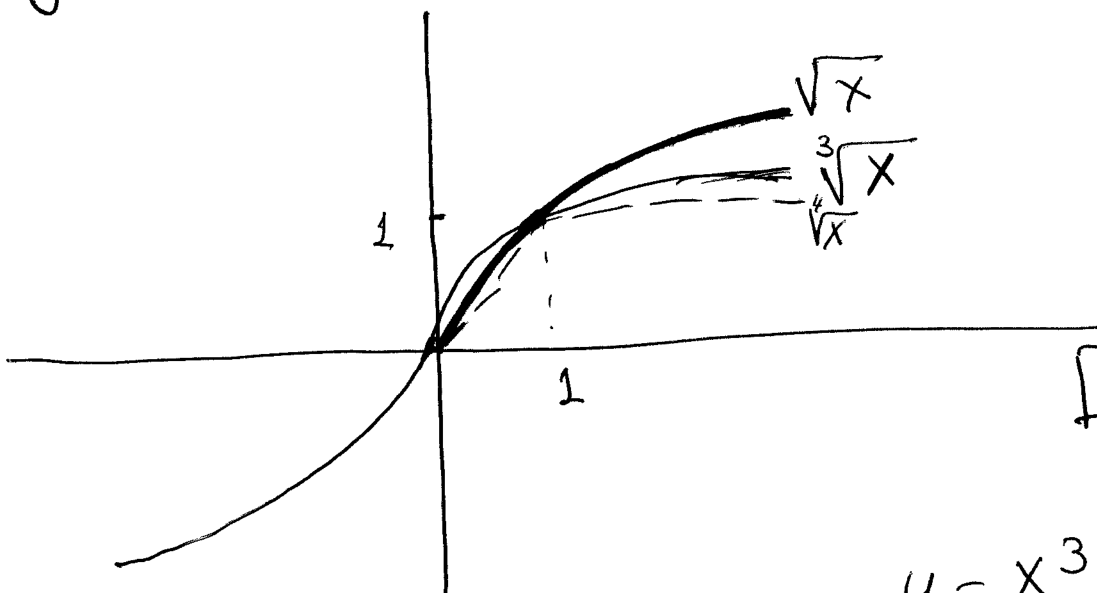
③ A **Power function** is of the form

$f(x) = x^r$, r is a real number.

$\boxed{r = \frac{1}{3}, \frac{1}{2}}$

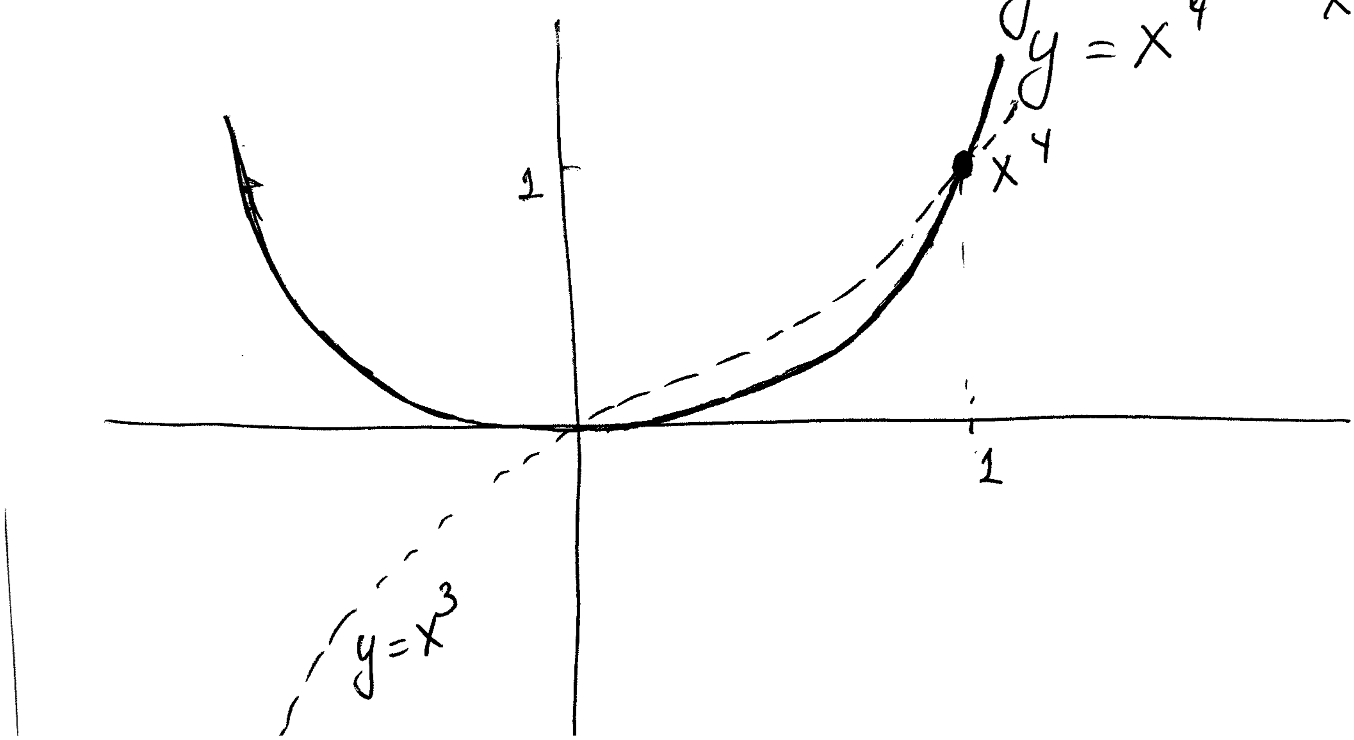
$y = x^{\frac{1}{3}} = \sqrt[3]{x}$, $x \in \mathbb{R}$

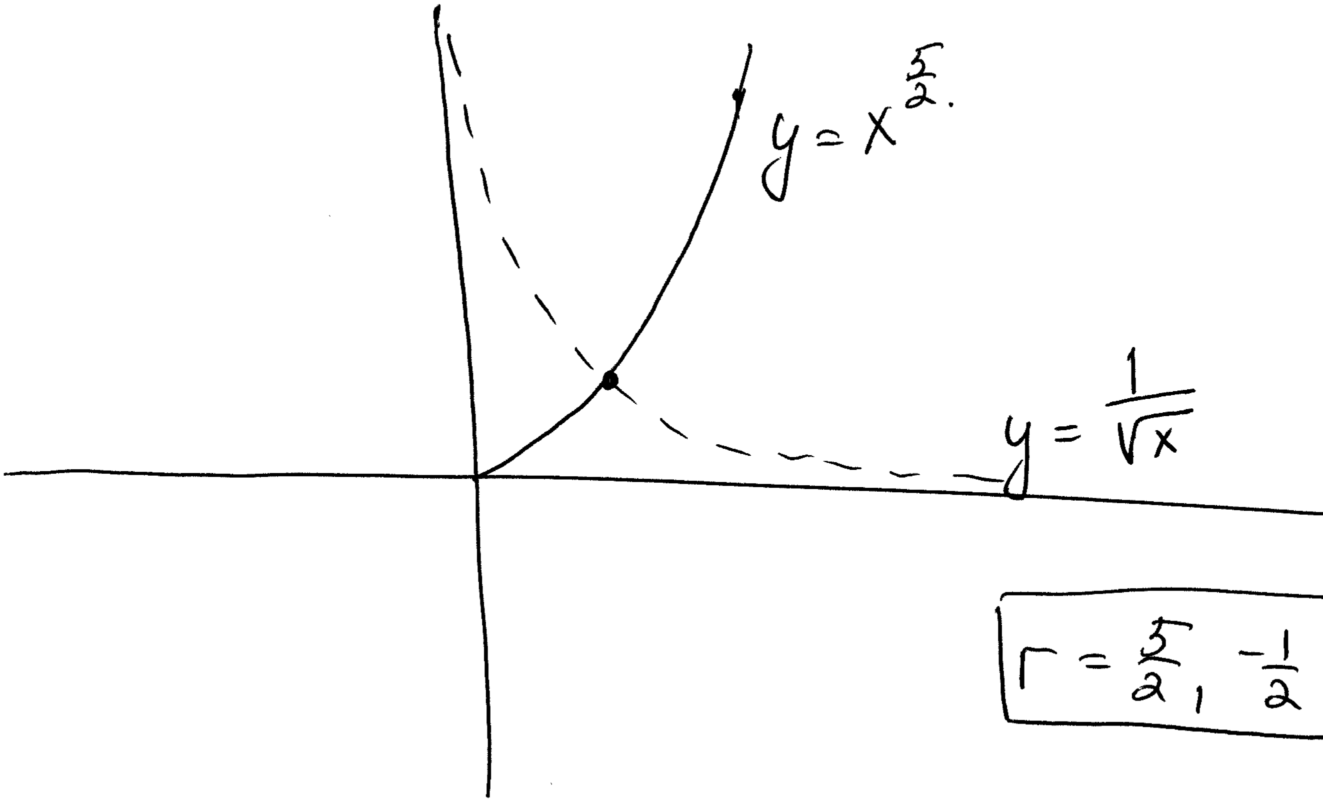
$y = x^{\frac{1}{2}} = \sqrt{x}$, $x \geq 0$



$\boxed{r = 3, 4}$

$y = x^3$
$y = x^4$ $x \in \mathbb{R}$.





$$\boxed{r = \frac{5}{2}, \; -\frac{1}{2}}$$

$$y = f(x) = x^{\frac{5}{2}} = \sqrt{x^5} \quad , \; x \geq 0$$

$$y = f(x) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} \quad , \; x > 0.$$

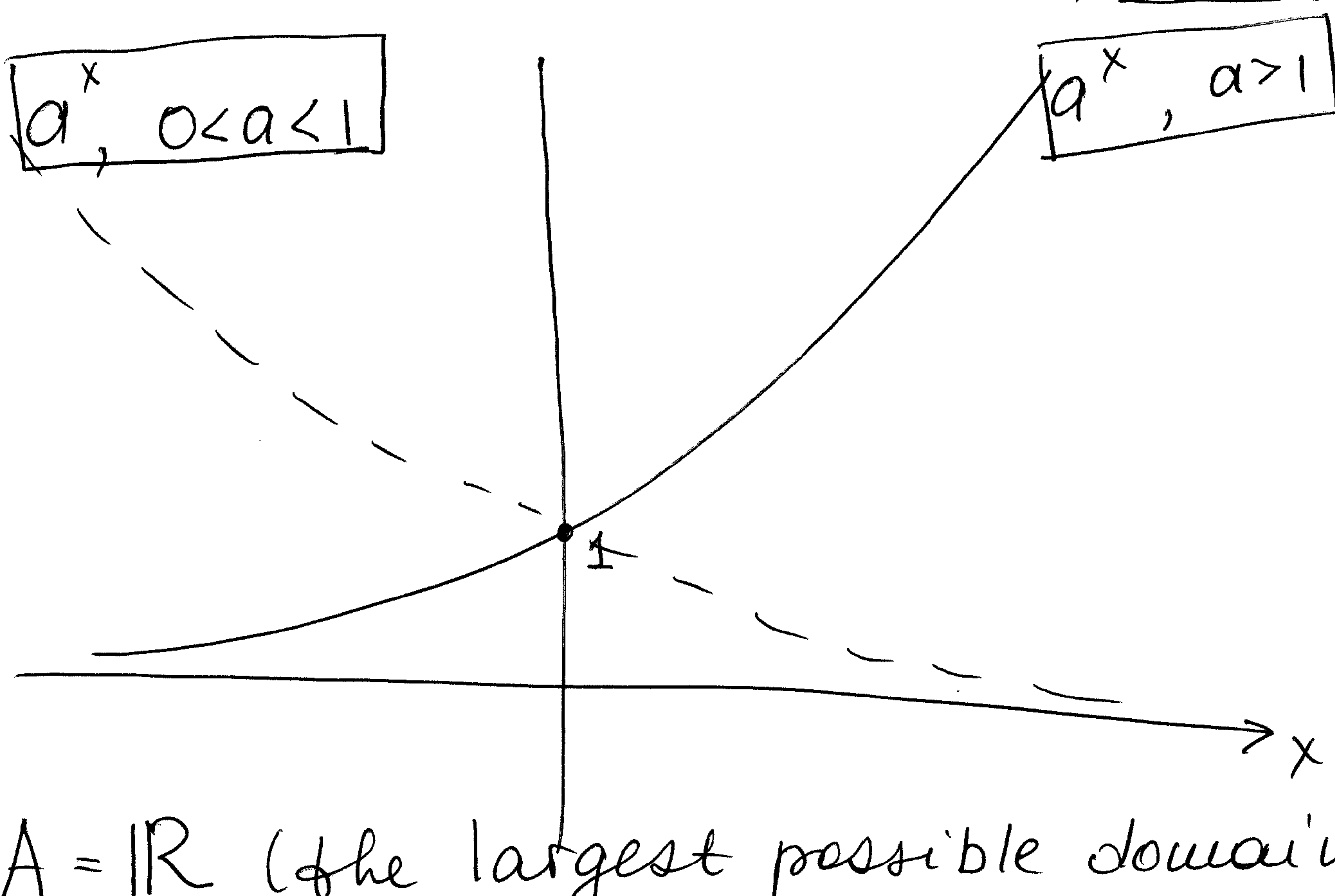
Properties of Power functions

$$\boxed{\begin{aligned} &x^r \cdot x^s = x^{r+s} \\ &(x^s)^r = x^{sr} \\ &\frac{x^r}{x^s} = x^{r-s} \\ &x^0 = 1 \end{aligned}}$$

FSM | 1 , 7

(4) The function f is an exponential function with base a if

$y = f(x) = a^x$, a is a positive constant, $\boxed{a \neq 1}$



$A = \mathbb{R}$ (the largest possible domain)

$B = (0, +\infty)$ (the range)

The Properties of exponentials

$\boxed{a^m \cdot a^n = a^{m+n}} \rightarrow 3^4 \cdot 3^5 = 3^{4+5} = 3^9$

$\boxed{\frac{a^m}{a^n} = a^{m-n}} \Rightarrow \frac{3^5}{3^4} = 3^{5-4} = 3^1$

$\boxed{a^{-m} = \frac{1}{a^m}} \Rightarrow 3^{-4} = \frac{1}{3^4}$

$3^4 = \frac{1}{3^{-4}}$

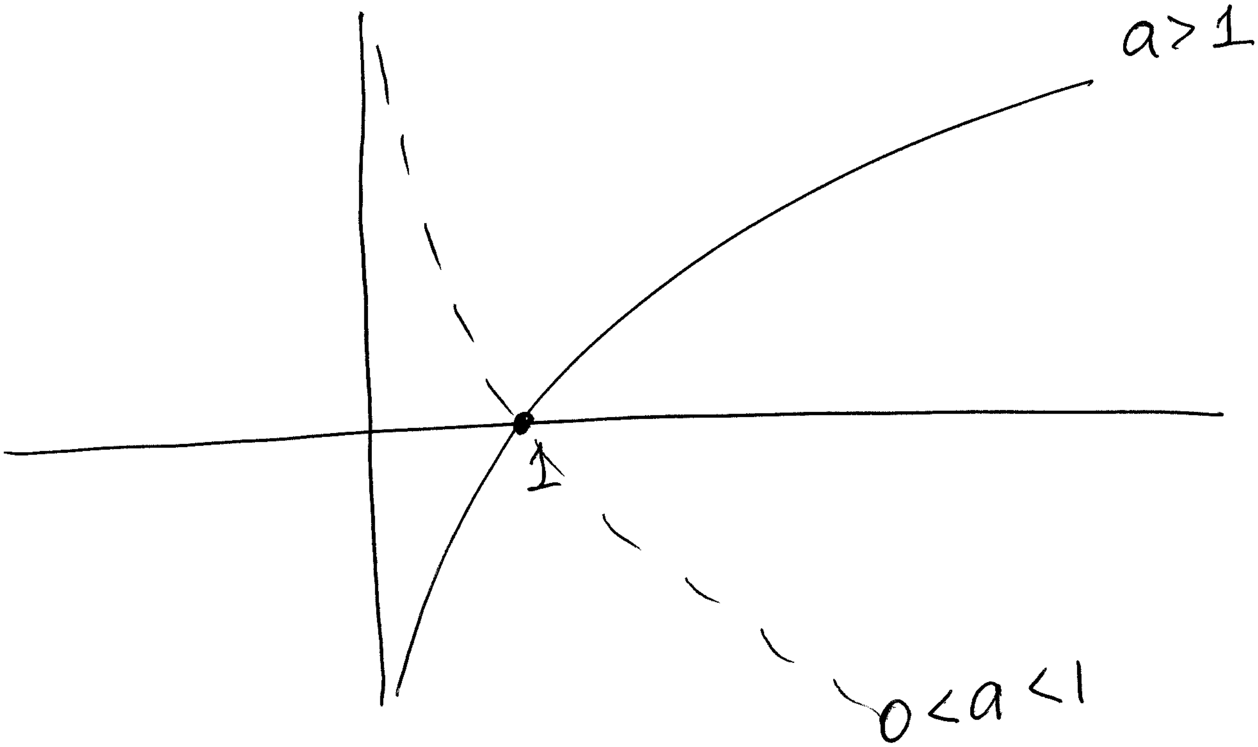
$\boxed{(a^m)^n = a^{m \cdot n}} \Rightarrow (3^4)^5 = 3^{20}$

$\boxed{a^0 = 1}$

5

The logarithmic function to base a is the function of the form:

$$y = f(x) = \log_a x$$



$A = \{x > 0\}$ (the domain)

$B = \mathbb{R}$ (the range)

The logarithm satisfies the following properties:

1.) $a^{\log_a x} = x \, , \quad x > 0$

2.) $\log_a a^x = x \, , \quad x \in \mathbb{R}$

3.) $\log_a (s \cdot t) = \log_a(s) + \log_a(t)$

4.) $\log_a \left(\frac{s}{t}\right) = \log_a(s) - \log_a(t)$

5.) $\log_a (s^r) = r \log_a(s)$

---

6 Trigonometric functions (next lecture)

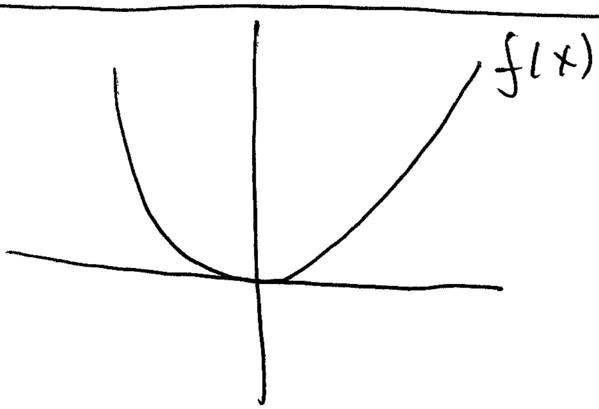
Transformation of functions (a tool for building new functions)

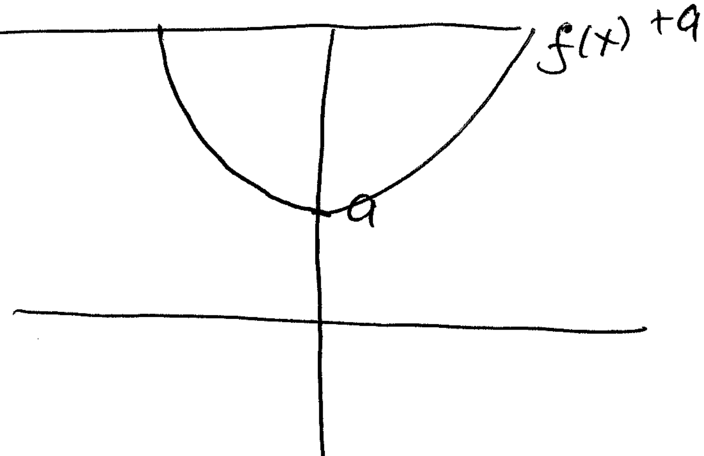
→ We have $y = f(x)$, then

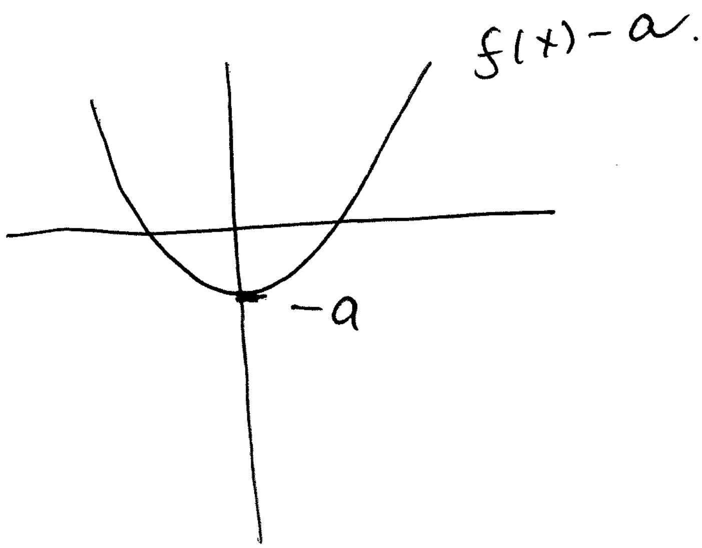
$y = f(x) + a$ is a <u>vertical translation.</u>

if $a > 0$, the graph of $f(x)$ is shifted up **a** units.

if $a < 0$, the graph of $f(x)$ is shifted down $|a|$ units.



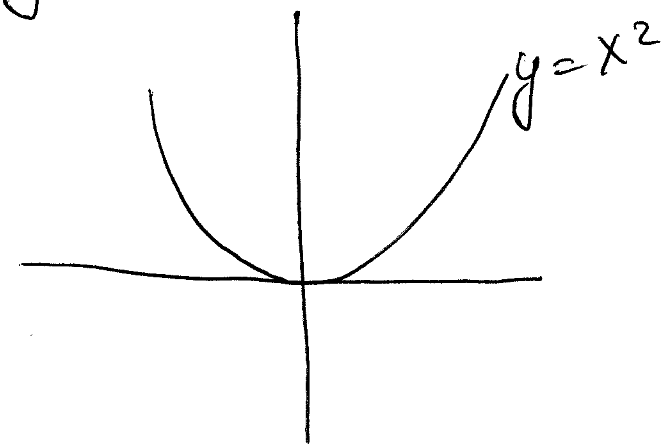


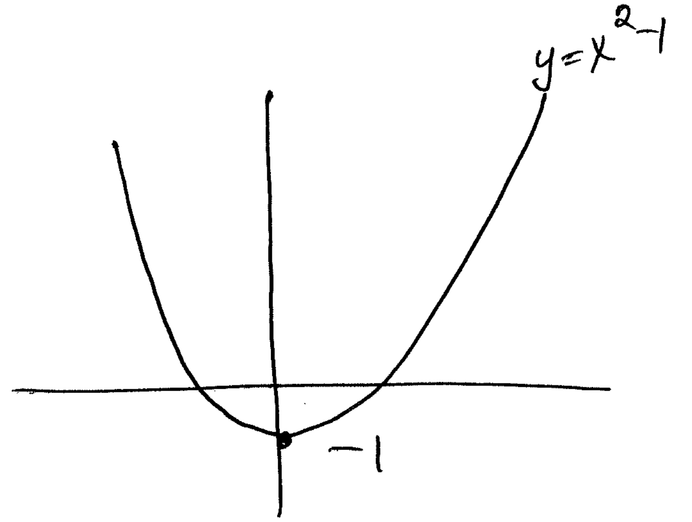


FSM 1 18

Find the domain and range of

$y = x^2 - 1$





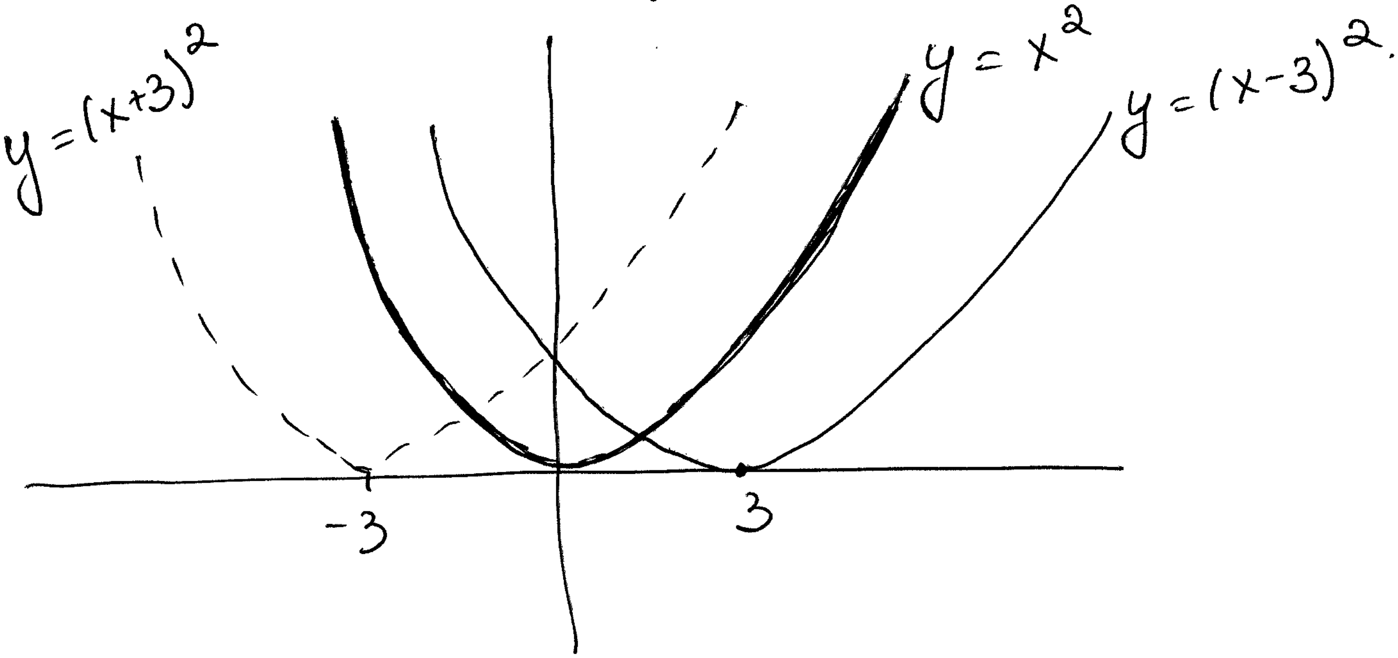
$A = \mathbb{R}$

$B = [-1, +\infty)$

→ The graph of $y = f(x+c)$ is a horizontal translation of $y = f(x)$

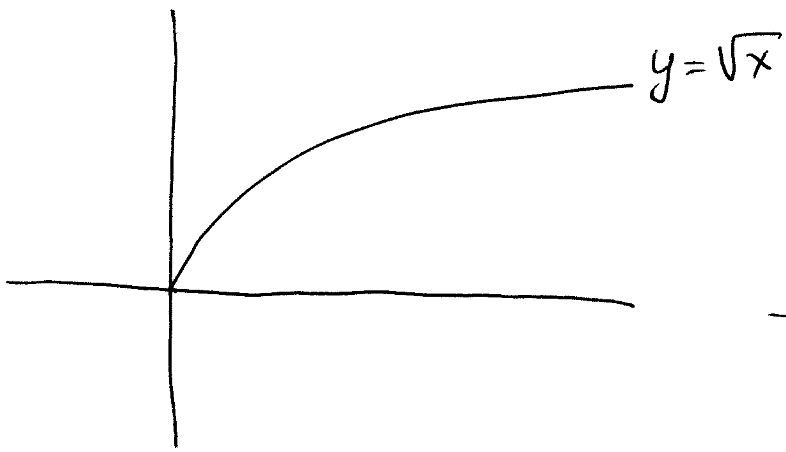
if $c > 0$, the graph of f is shifted c units to the left ($\leftarrow$).

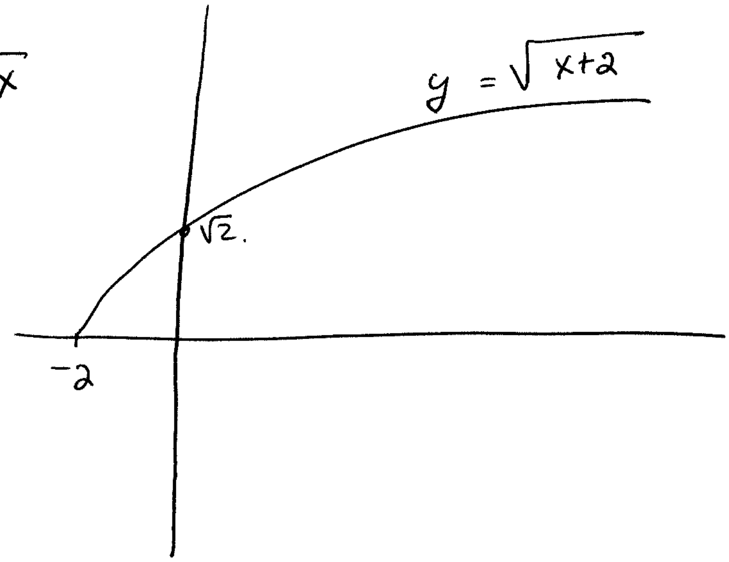
if $c < 0$, the graph of f is shifted c units to the right ($\longrightarrow$)

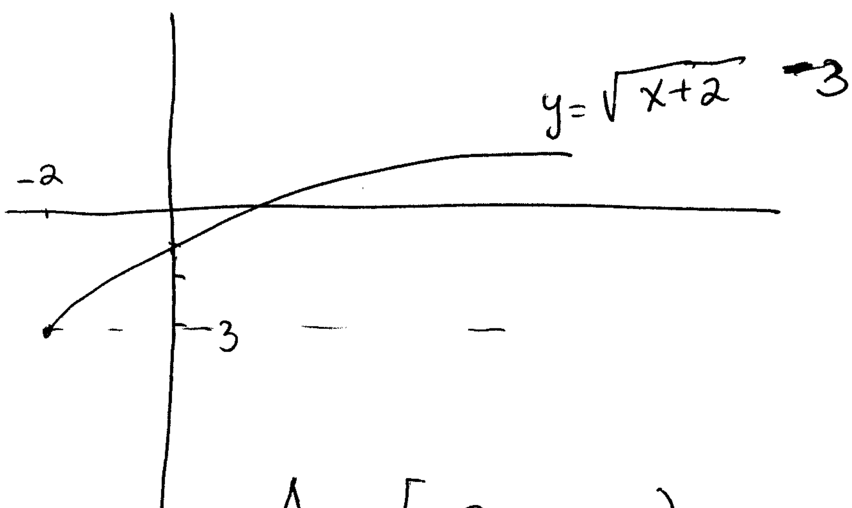


FSM 19.

$y = \sqrt{x+2} - 3$







$A = [-2, +\infty)$

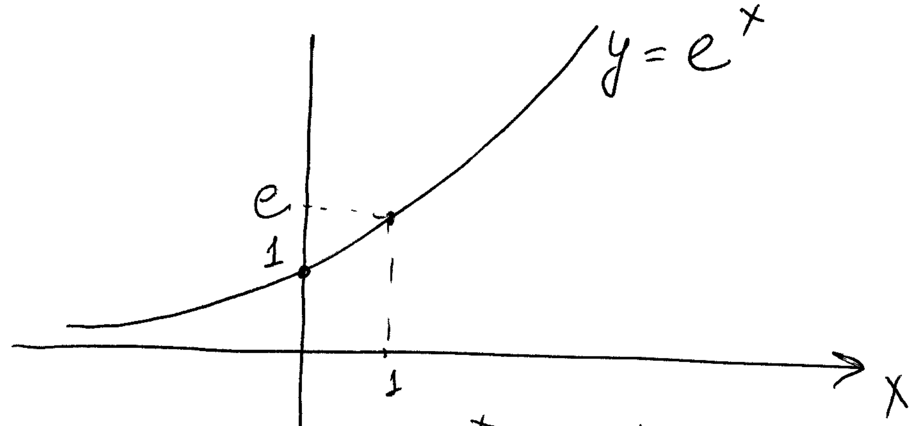
$x+2 \geq 0, \ x \geq -2.$

$B = [-3, +\infty).$

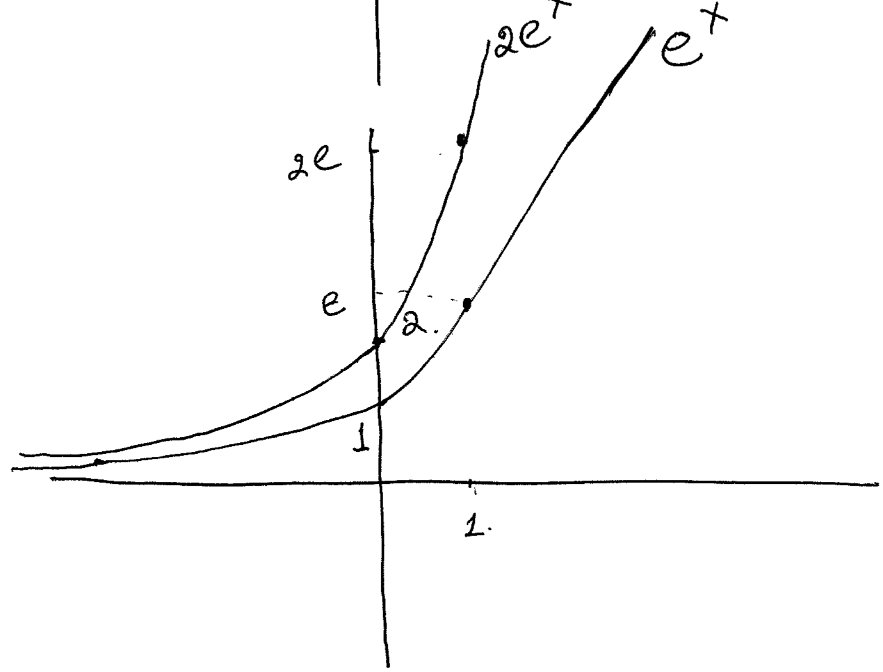
→ $\boxed{y = C \cdot f(x)}$, $c > 1$ – stretch the graph of $f(x)$ vertically by a factor of C.

if $0 < C < 1$ – compress $f(x)$ vertically by a factor of $\frac{1}{C}$

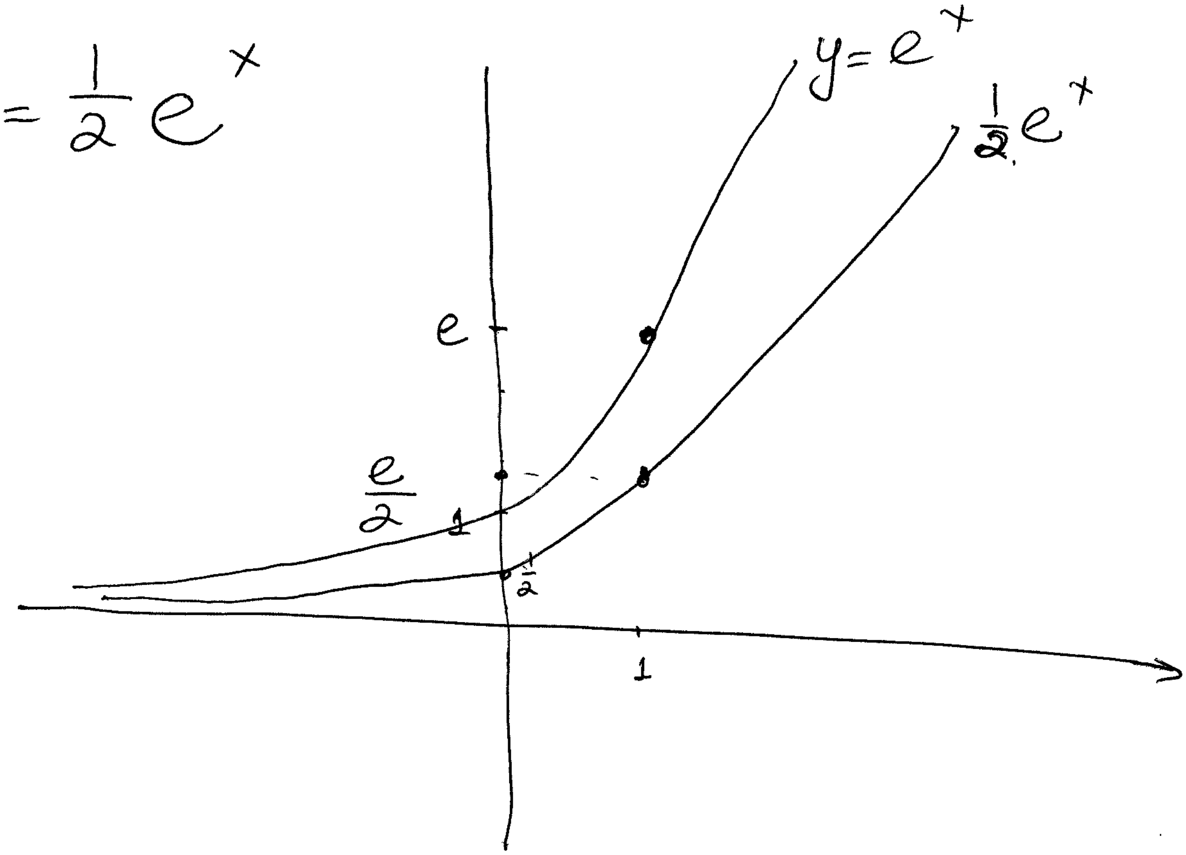
$y = f(x) = e^x$



$y = 2e^x$,

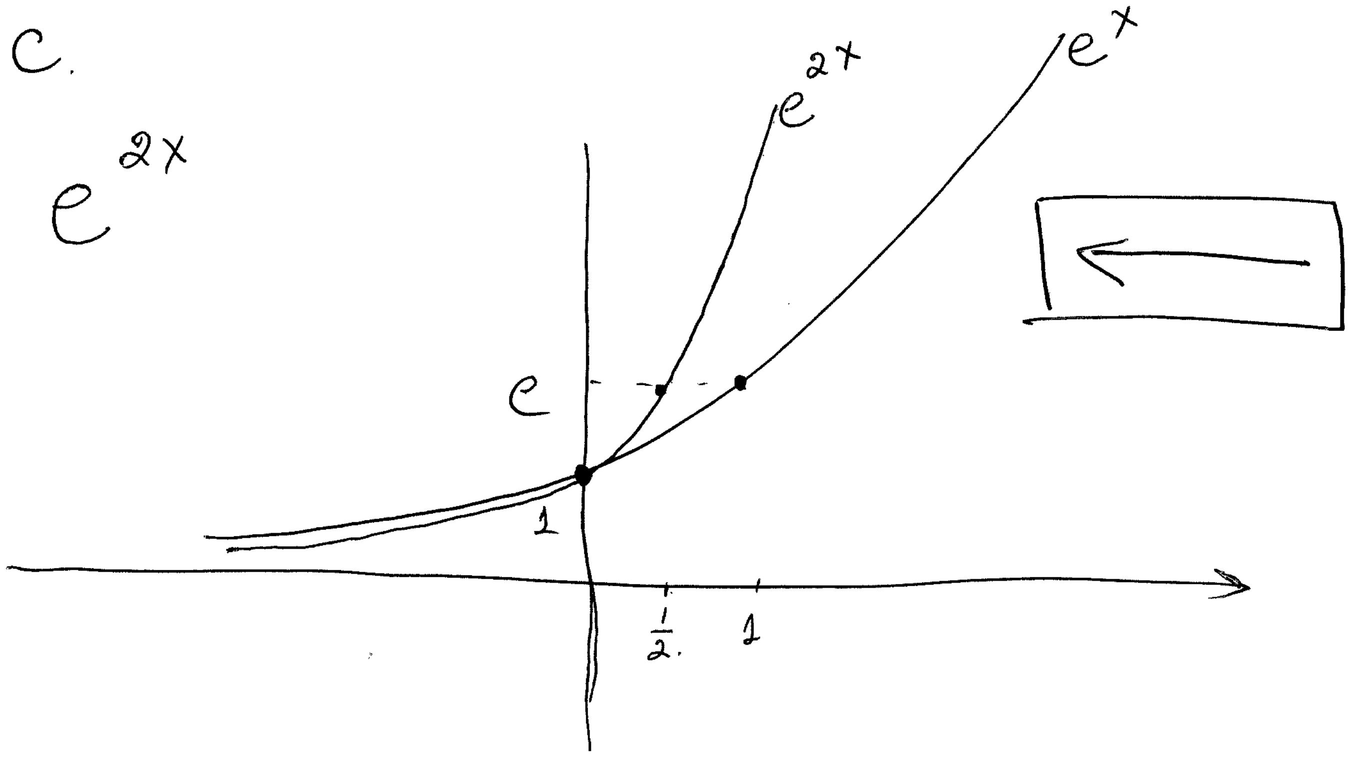


$y = \frac{1}{2} e^{x}$



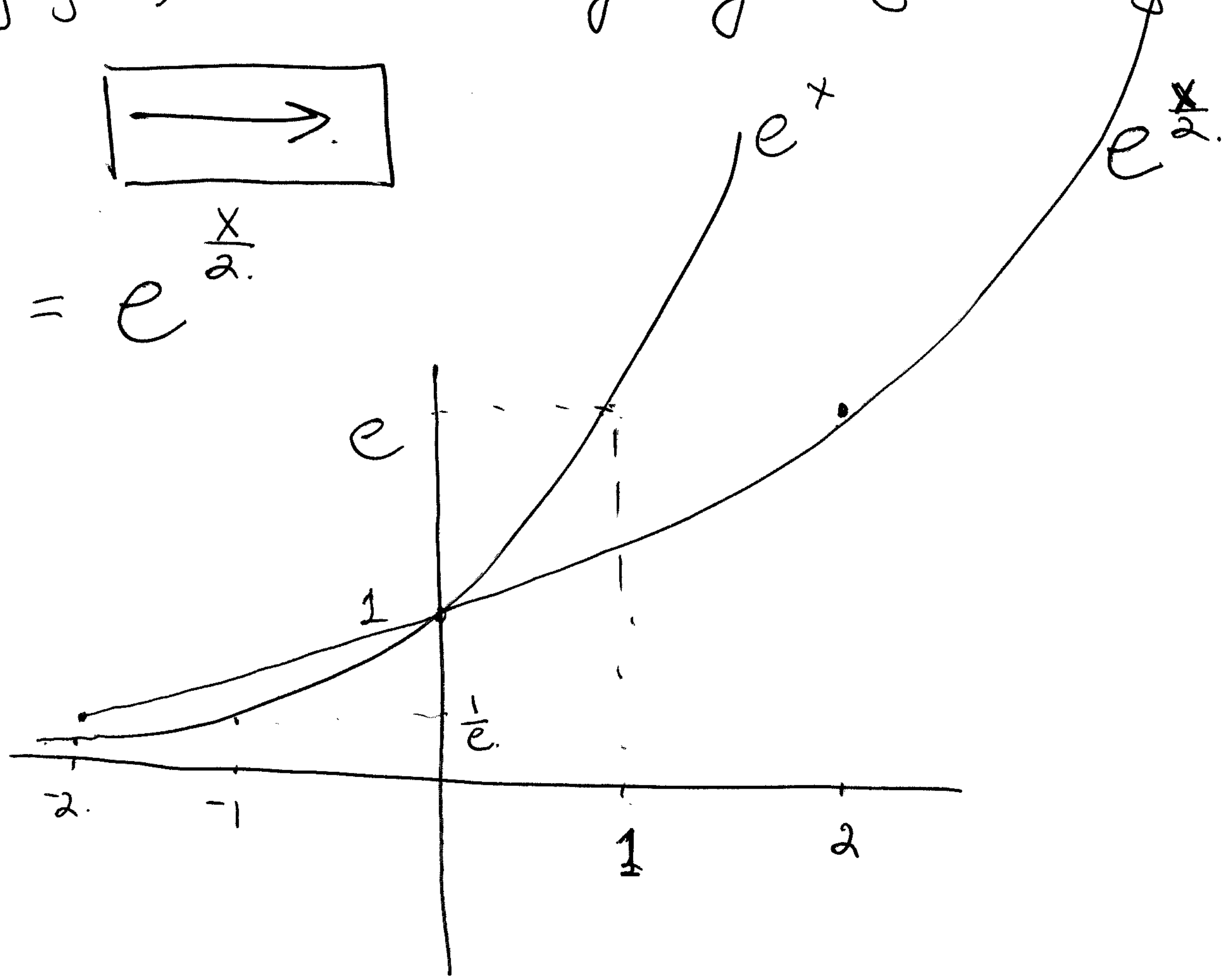
$\rightarrow y = f(c \cdot x)$, $c > 1$ – compress the graph of $f(x)$ horizontally by a factor of c.

$y = e^{2x}$



if $0<c<1$ – we stretch a graph of $f(x)$ horizontally by a factor of $\frac{1}{c}$.

$$y = e^{\frac{x}{2}}$$

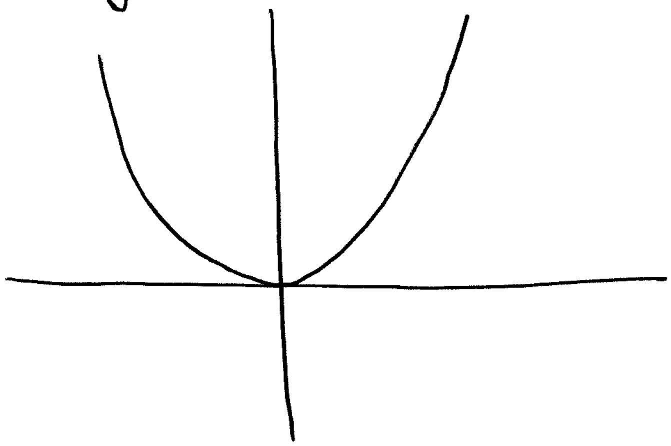


→ $y = -f(x)$ – reflect the graph of $f(x)$ with respect to the x-axis.

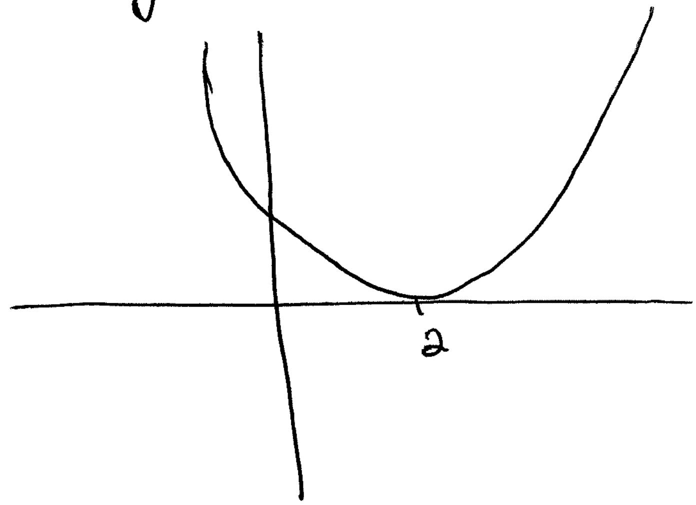
FSM 1 12

$$y = -(x-2)^2 + 1.$$

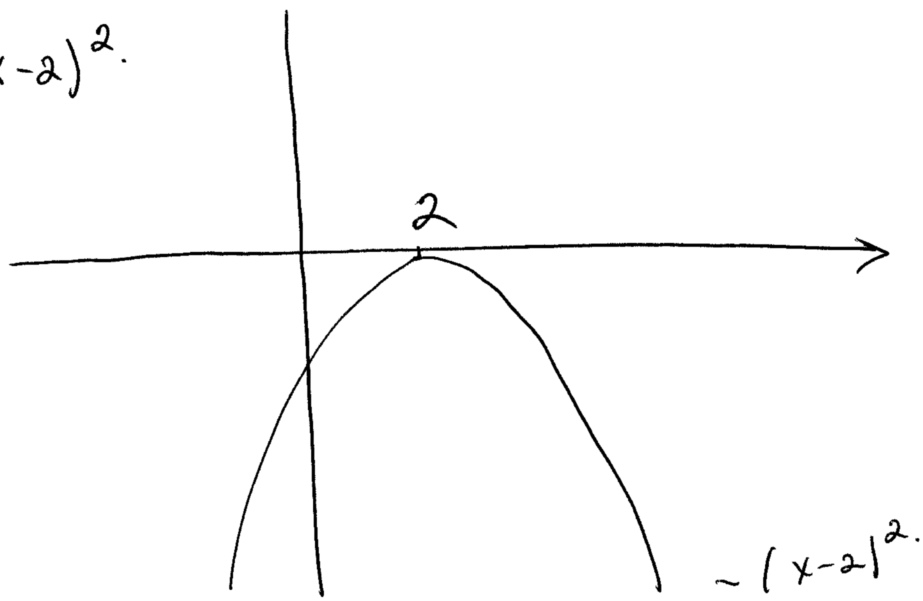
$y = x^2$



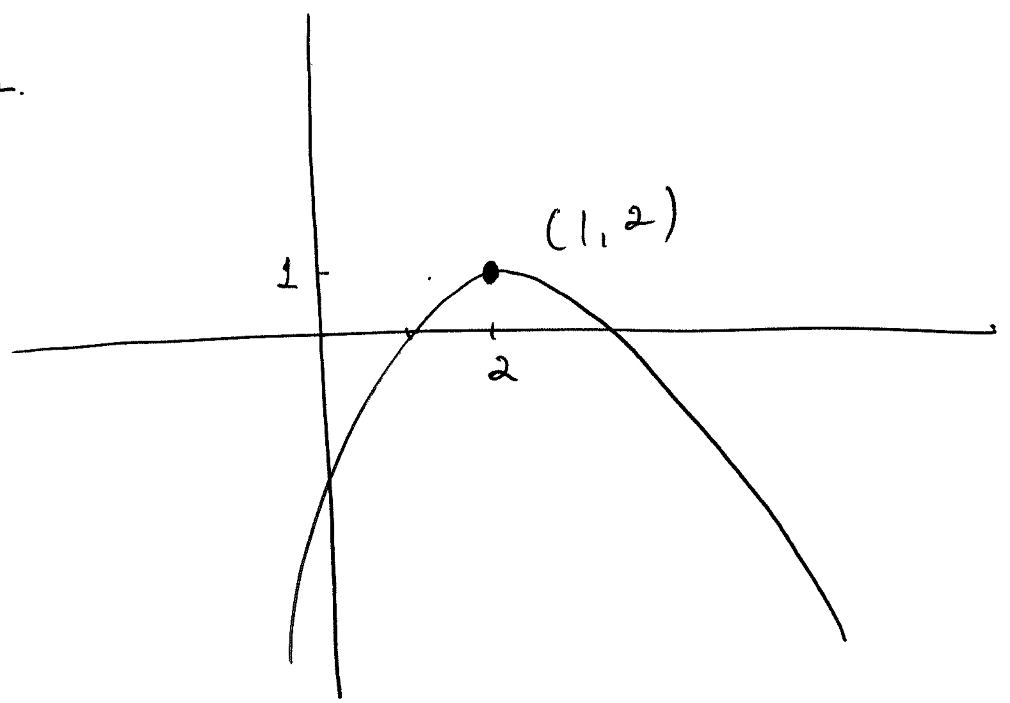
$y = (x-2)^2.$



$y = -(x-2)^2.$

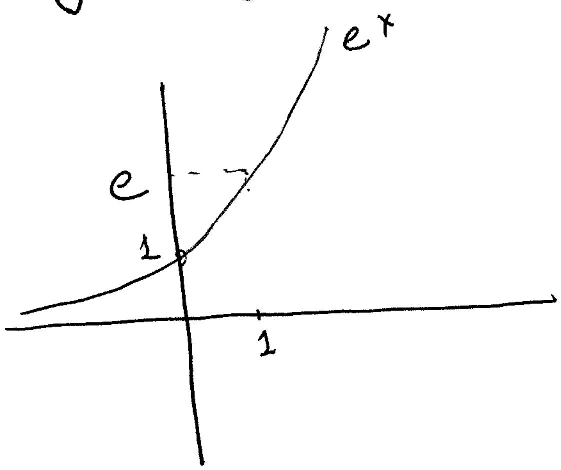


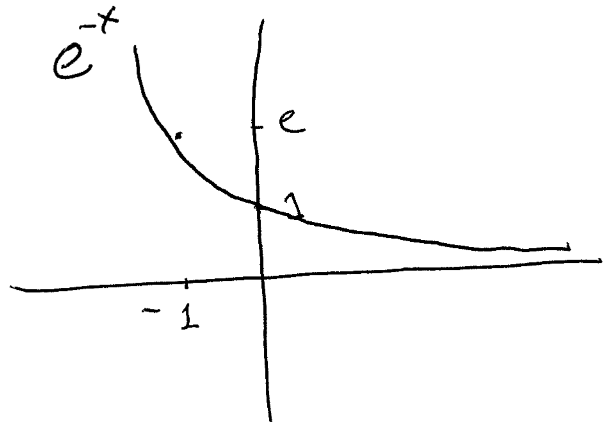
$y = -(x-2)^2+1.$

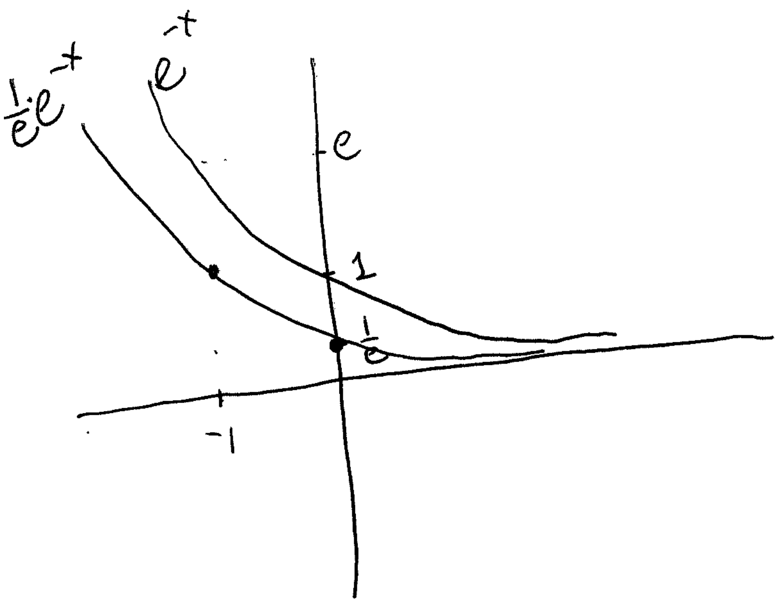


→ $y = f(-x)$ – reflect the graph of $f(x)$ with respect to the y-axis.

$$y = \frac{1}{e^{x+1}} = e^{-(x+1)} = e^{-x-1} = e^{-x} \cdot e^{-1} = \frac{1}{e} e^{-x}$$



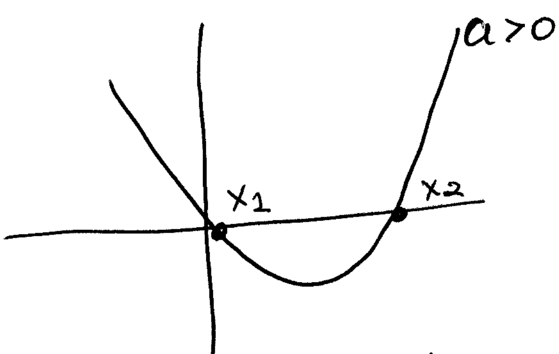


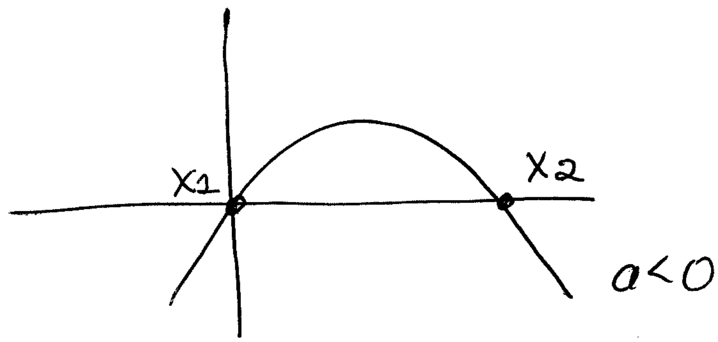


$$y = ax^2 + bx + c.$$

The following universal formulae helps you to find the roots of a quadratic polynomial.

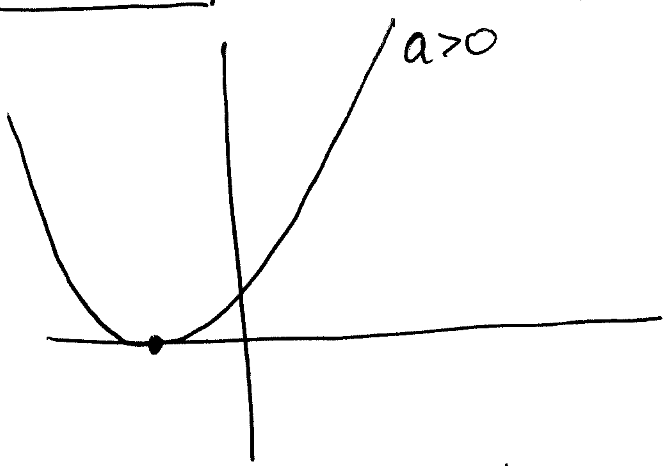
Case 1 $\quad D = b^2 - 4ac > 0$

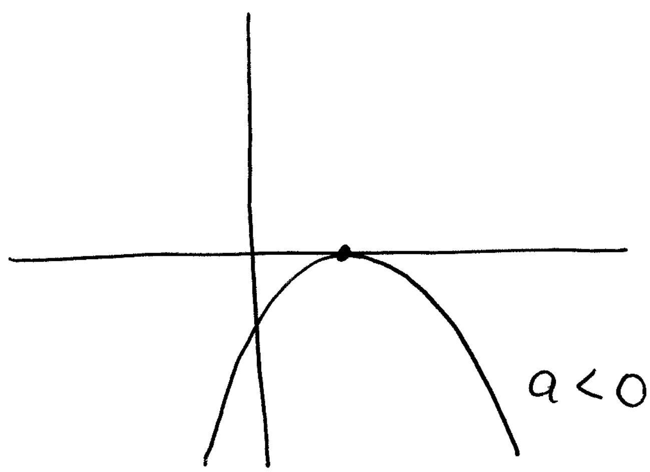




$$x_1 = \frac{-b + \sqrt{D}}{2a} \; ; \; x_2 = \frac{-b - \sqrt{D}}{2a}$$

Case 2 $\quad D = b^2 - 4ac = 0$

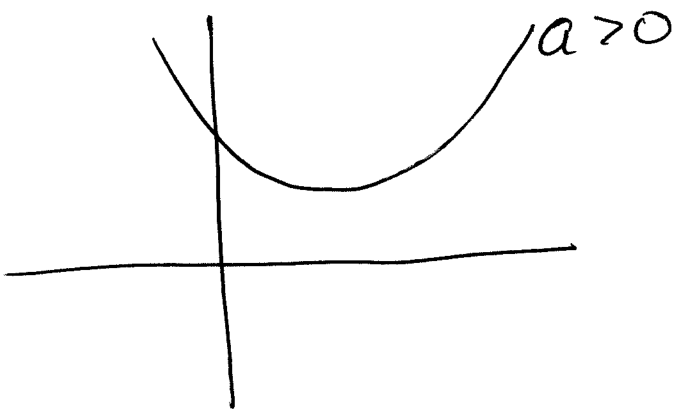


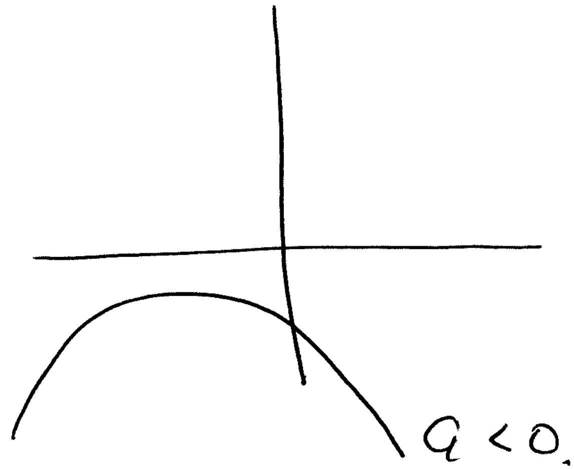


$$x_1 = x_2 = \frac{-b}{2a}$$

$\boxed{\text{Case 3}}$ $\quad D < 0$

no real roots (only complex roots)





$$\boxed{ax^2 + bx + c = a(x - x_1)(x - x_2)}$$

Factoring forumulas

$(x+y)(x-y) = x^2 - y^2$

$(x \pm y)^2 = x^2 \pm 2xy + y^2$

$x^3 \mp y^3 = (x \mp y)(x^2 \pm xy + y^2)$

Rationalize the denominator

FSM 1 8 - 9

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})} =$$

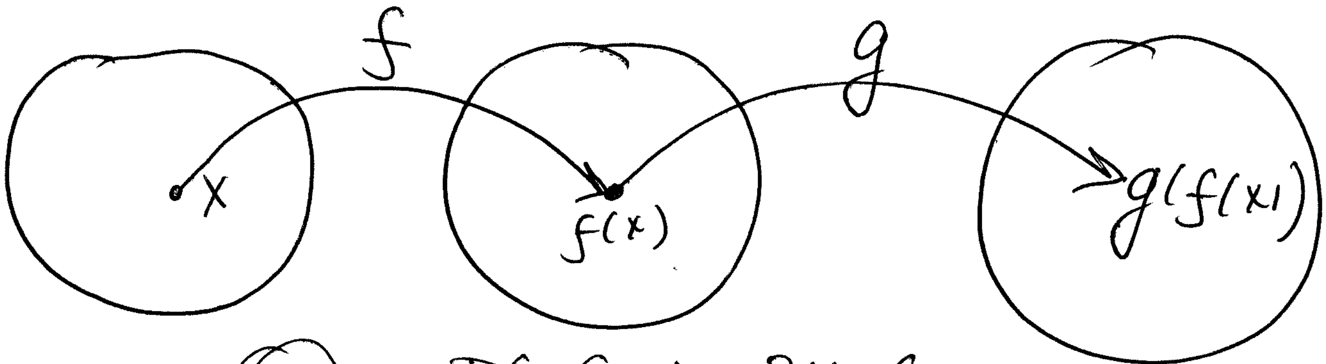
$$= \frac{\sqrt{a}-\sqrt{b}}{a-b}.$$

Example 8

$$\frac{1}{\sqrt{8}-5} = \frac{\sqrt{8}+5}{(\sqrt{8}-5)(\sqrt{8}+5)} =$$

$$= \frac{\sqrt{8}+5}{8-25} = -\frac{(\sqrt{8}+5)}{17}$$

The composite function $g \circ f =$
$= g(f(x))$



<u>FSMI</u> 15 If $f(x) = 3x - 6$,

find $f(3x-6)$.

$$f \circ f = f(f(x)) = 3(3x-6) - 6 =$$
$$= 9x - 24.$$