1. a) Define what it means for a function $f : \mathbf{R}^n \to \mathbf{R}^n$ to be an *isometry.*

b) Define what if means for an $n \times n$ matrix A to be an *orthogonal* matrix.

c) Give the general formula for an isometry $f : \mathbf{R}^n \to \mathbf{R}^n$. (Give all details for the symbols you use.)

d) Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbf{R}^n$ and a vector $v \in \mathbf{R}^n$ is orthogonal to every vector in $\mathcal{B}$. Prove that $v = 0$.

(1) a) An isometry f must satisfy $\|f(u) - f(v)\| = \|u - v\|$, $\forall u, v \in \mathbb{R}$

(1) b) An n×n matrix A is orthogonal if $A^tA = I_n$

(2) c) Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ has a formula (1)

$\forall v \in \mathbb{R}^n$, $f(v) = Av + b$ (1), for some $A \in O(n)$ and some (fixed) $b \in \mathbb{R}^n$

(2) d) Since $\mathcal{B}$ is a basis, $v = c_1v_1 + \cdots + c_nv_n$ for some (unique) scalars $c_1, \ldots, c_n \in \mathbb{R}$. Then,

$v \cdot v = (c_1v_1 + \cdots + c_nv_n) \cdot v = c_1 v_1 \cdot v + c_2 v_2 \cdot v + \cdots + c_n v_n \cdot v$

$= c_1 0 + c_2 0 + \cdots + c_n 0$

$= 0.$

Hence $\|v\|^2 = 0$ and so $v = 0$.

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2. Let $\theta \in (0, 2\pi)$, $\alpha = \cos\theta$, $\beta = \sin\theta$ and $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(v) = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} v$, for all $v \in \mathbf{R}^2$.

(1) a) Show that f is an isometry of $\mathbf{R}^2$. *(You may use a theorem but you must check all hypotheses.)*

Now let $L = \{(x, y) \in \mathbf{R}^2 \mid (\alpha - 1)x + \beta y = 0\}$.

(2) b) Show that L is a line through the origin in $\mathbf{R}^2$. *(Hint: Don't work hard: there are only two simple things to check here.)*

(3) c) Find the formula for reflection in the line L.
(Hint: Remember that $\alpha^2 + \beta^2 = 1$ and that $1 - \alpha^2 = (1-\alpha)(1+\alpha)$).

(1) a) The map f is an isometry since $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}^t \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$

$= \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2+\beta^2 & 0 \\ 0 & \alpha^2+\beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

(2) b) L will be a line as long as $(\alpha-1, \beta) \neq (0,0)$. (1) But $\alpha = 1 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = 2n\pi$ $(n \in \mathbb{Z})$, none of which values of theta lie in $(0, 2\pi)$. Hence L is a line. It passes through the origin (1) because $(\alpha-1)0 + \beta 0 = 0$.

c) We know that $R_L(x,y) = (x,y) - 2\dfrac{[(\alpha-1)x + \beta y]}{(\alpha-1)^2+\beta^2}(\alpha-1, \beta)$ (1)

Now, $(\alpha-1)^2 + \beta^2 = \alpha^2 - 2\alpha + 1 + \beta^2 = 2 - 2\alpha$ so

$R_L(x,y) = (x,y) - \dfrac{[(\alpha-1)x + \beta y]}{1-\alpha}(\alpha-1, \beta)$

The first component of the RHS is

$x + [(\alpha-1)x + \beta y] \quad) = \alpha x + \beta y$

2c) cont. The second component of RHS is

$$y - [(\alpha-1)x + \beta y]\frac{\beta}{(1-\alpha)} = y + \beta x - \frac{\beta^2}{1-\alpha} \cdot y$$

But $\beta^2 = 1-\alpha^2 = (1-\alpha)(1+\alpha)$, so this is

$$y + \beta x - \frac{(1-\alpha)(1+\alpha)}{(1-\alpha)} y = y + \beta x - y - \alpha y$$

$$= \beta x - \alpha y.$$

(2) Hence $R_L(x,y) = (\alpha x + \beta y, \beta x - \alpha y)$ $(\overset{!}{=} f(x,y))$.

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3. Suppose $a \in \mathbf{R}^n$, $a \neq 0$, $b \in \mathbf{R}$ and H is the hyperplane

$$H = \{v \in \mathbf{R}^n \mid a \cdot v = b\}.$$

(3) a) Show, using the formula on the title page, that $H = \{v \in \mathbf{R}^n \mid R_H(v) = v\}$. (Careful: It's not enough to show that $H \subseteq \{v \in \mathbf{R}^n \mid R_H(v) = v\}$)

(3) b) Prove, using the formula on the title page, that $R_H(R_H(v)) = v$ for all $v \in \mathbf{R}^n$.

a) Since $R_H(v) = v - \frac{2(a \cdot v - b)}{\|a\|^2} a$, (1) $R_H(v) = v$

$\Leftrightarrow -\frac{2(a \cdot v - b)}{\|a\|^2} a = 0$ (1) $\Leftrightarrow a \cdot v - b = 0 \Leftrightarrow v \in H$ (1)

(since $\frac{2a}{\|a\|^2} \neq 0$)

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b) $R_H(R_H(v)) = R_H\left(v - \frac{2(a \cdot v - b)}{\|a\|^2} a\right)$

(3)

$= v - \frac{2(a \cdot v - b)}{\|a\|^2} a - \frac{2}{\|a\|^2} a \left[a \cdot \left(v - \frac{2(a \cdot v - b)}{\|a\|^2}\right) a\right]$

$= v - \frac{2(a \cdot v - b)a}{\|a\|^2} - \frac{2a}{\|a\|^2}\left(a \cdot v - 2(a \cdot v - b)\right)$

$= v - \frac{2(a \cdot v - b)a}{\|a\|^2} - \frac{2a}{\|a\|^2}(-a \cdot v + b)$

$= v$, as required. 222

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4. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(x, y) = (-y, -x + 4)$.

(1) a) Find a 2×2 matrix A and $b \in \mathbf{R}^2$, and write the formula for f as $f(v) = Av + b$. Use this to show that f is an isometry.

(1) b) Is f orientation preserving or orientation reversing?

(2) c) Find the formula for $f(f(x, y))$.

(2) d) Is f a translation, a rotation, or a reflection?

a) Note that $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. So set

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

Since $A^t A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, f is an isometry

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b) Since $\det A = -1$, f is orientation reversing

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c) $f(f(v)) = f(Av + b) = A(Av + b) + b$

$= A^2 v + Ab + b = v + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$= v + \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = v + \begin{bmatrix} -4 \\ 4 \end{bmatrix}$; so $f(f(x,y)) = (x-4, y+4)$.

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d) Since f is orientation reversing it cannot be either a translation or a reflection. (1) On the other hand, since $f \circ f \neq id_{\mathbb{R}^2}$, f is not a reflection either (1) So it is none of the isometries listed

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5. (This is a bonus question: Do not attempt it until you have completed and checked all parts of questions 1-4)

(1) a) Show that the composition of two isometries is an isometry.

(3) b) Show that any isometry $f : \mathbf{R}^n \to \mathbf{R}^n$ has an inverse, that is, that there is a function $g : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(g(P)) = P$ and $g(f(P)) = P$ for all $P \in \mathbf{R}^n$.

(2) c) Show that the inverse of an isometry *is itself* an isometry.

a) Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be isometries, and $u, v \in \mathbb{R}^n$.

Then $\|f(g(u)) - f(g(v))\| = \|g(u) - g(v)\|$ (since f is an isom.)

$= \|u - v\|$ (since g is an isom.)

Hence $f \circ g$ is an isometry.

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b) Write $f(v) = Av + b$ for $A \in O(n)$, $b \in \mathbb{R}^n$.

Then $v = A^{-1}(f(v) - b) = A^{-1}f(v) - A^{-1}b$. Set $g(v) = A^{-1}v - A^{-1}b$. Then $f(g(v)) = f(A^{-1}v - A^{-1}b)$ $= A(A^{-1}v - A^{-1}b) + b = v - b + b = v$; moreover $gf(v) = g(Av + b) = A^{-1}(Av + b) - A^{-1}b) = v + A^{-1}b - A^{-1}b = v$.

Hence g is an inverse for f

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c) By (b) $g(v) = A^{-1}v - A^{-1}b$. Since $A \in O(n)$, $A^{-1} = A^t$, and $(A^t)^t A^t = AA^t = I_n$. Hence g is also an isometry