ANSWERS
Version A

Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

DISCRETE MATHEMATICS FOR COMPUTING — PROFESSOR: PHILIP SCOTT

MAT1348A — Test 5 — Tuesday, March 28, 2017

INSTRUCTIONS

- Clearly write your name and student number on this test, and **sign it** below to confirm that you will read and follow these **instructions**:
- This is a 75-minute **closed-book** test. **No notes.** **No calculators.**
- The exam consists of 7 questions. The total number of points possible is 34 points. The last page 9 is a worksheet: **please do not detach**.
- Questions 1–4 are **short-answer**. Write the final answer in the appropriate answer box. You do not need to show any work.
- Questions 5–7 are **long-answer**. To receive full marks, your solution/proof must be complete, correct, and show all relevant details. **Carefully state definitions or any required notions**.
- Read all questions carefully and be sure to follow the instructions for the individual problems.
- For rough work or additional work space, you may use the backs of pages.
  The last page (Page 9) may be used for scrap work. Please do not detach Page 9.
  Do not use any of your own scrap paper.
- You must use **proper mathematical notation and terminology**.

Cellular phones, computers, calculators, tablets, unauthorized electronic devices, and course notes are **not allowed** during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read, understand, and will comply with the above instructions.**

FAMILY NAME:	STUDENT NUMBER:
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Please rewrite your name and student number here: *(do not write in the table of points below)*

FAMILY NAME:	FIRST NAME:
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Table of Points for marking purposes.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
Maximum points	2 pts	4 pts	6 pts	6 pts	5 pts	5 pts	6pts	34 points
Marks obtained								

SHORT-ANSWER QUESTIONS.

Write your final answer in the answer box.

(2pts) **Q1.** How many binary strings of length 8 contain exactly 3 zeros? *Fully evaluate your answer*

$$\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{3\cdot 2}\cdot \cancel{5!}} = 56$$

(4 pts) **Q2.** Fully evaluate the following expressions and write the answer in the box:

1a) $P(10, 3) =$ 720

$$P(10,3) = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = 90 \cdot 8 = 720$$

1b) $C(10, 7) =$

$$C(10,7) = \binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4}}{\cancel{3} \cdot \cancel{2}} = 120$$

(6 pts) **Q3.** Let $A = \{1,2,3\}$ and $B = \{1,2,3,4,5,6\}$. Determine the following.
Fully evaluate your answers.

(a) The number of functions $f : A \to B$ that are **injective** (one-one).

$$6 \cdot 5 \cdot 4 = 30 \cdot 4 = 120$$

$1 \mapsto 6$ possible outputs
$2 \mapsto 5$ possible outputs
$3 \mapsto 4$ possible outputs

(b) The number of functions $f : A \to A$ that are **not injective**.

$$|A^A| - \#\text{inj } f^{ns} = 3^3 - 3! = 27 - 6 = 21$$

(c) The number of subsets of B of cardinality 4 that contain the element 2:

See below:

$$\binom{6-1}{4-1} = \binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2} = 10$$

(6 pts) **Q4.**

(a) State the Generalized Pigeonhole Principle (GPHP)

Generalized Pigeonhole Principle:

If N objects are placed into k boxes, then at least one box contains at least $\lceil N/k \rceil$ objects.

(b) A box contains 10 red, 10 blue, 10 yellow, and 10 green marbles. Blindly you pull out n marbles. What is the smallest value of n to guarantee that:

(i) among the n marbles, there are at least 3 marbles of *same* colour?

There are 4 boxes (= colours). Solve $\lceil N/4 \rceil = 3$.

Then $\boxed{N = 9}$; since $\lceil \frac{N}{4} \rceil = \lceil \frac{9}{4} \rceil = \lceil 2.25 \rceil = 3$

(whereas $\lceil \frac{8}{4} \rceil = 2$.)

(ii) among the n marbles, there are at least 3 marbles of *different* colours.

Worst case: pick $10 + 10 = 20$ all from two colors so if you pick 21 you must have 3 distinct colours (at least).

Long-Answer Questions.

Detailed solutions are required. **Give precise definitions of all relevant notions**

(5 pts) **Q5.** (Typo corrected at the beginning of test)

How many strings of length 9 of decimal digits(i.e. using symbols from $\{0, 1, 2, \ldots, 9\}$) start with "987" ~~and~~ or end with "10" (*This is inclusive or.*).

Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.

Number of such strings:

$$10^6 + 10^7 - 10^4$$

Carefully explain your reasoning (just getting an answer, even if correct, is not sufficient for full credit.):

Let A = strings $9\ 8\ 7\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{6 slots empty}}$ strings starting with 987

$|A| = 10^6$

Let B = strings $\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{7 slots empty}}\ 1\ 0$ strings ending in 10.

$|B| = 10^7$

So $A \cap B$ = strings $9\ 8\ 7\ \underbrace{\_\ \_\ \_\ \_}_{\text{4 slots empty}}\ 1\ 0$ } strings starting 987 & ending 10

$|A \cap B| = 10^4$

$\therefore |A \cup B| = |A| + |B| - |A \cap B|$

$= 10^6 + 10^7 - 10^4$

(5 pts) **Q6.** *Your answer to this question may include unevaluated factorials, binomial coefficients, powers, products, or sums. Explain your reasoning in detail.*

A committee of 7 people is to be chosen from a group of 10 mathematicians and 9 computer scientists. Determine the number of such committees that contain:

(a) **Exactly** 3 mathematicians.

$$\binom{10}{3}\binom{9}{4}$$

choose mathematicians

choose computer scientists

uses product rule

(b) **At most** 3 mathematicians.

$$\binom{10}{0}\cdot\binom{9}{7}+\binom{10}{1}\binom{9}{6}+\binom{10}{2}\binom{9}{5}+\binom{10}{3}\binom{9}{4}$$

$$=\sum_{i=0}^{3}\binom{10}{i}\binom{9}{7-i}$$

Either 0 math & 7 CS or 1 math, 6 CS
or 2 math & 5 CS or 3 math & 4 CS profs.

Then use product rule for forming each $\binom{10}{i}\binom{9}{7-i}$ and sum rule for all possible $i=0$ to 3.

(6 pts) **Q7.** The goal of this problem is to prove the following by mathematical induction.

$$\textit{For all integers } n \geq 1, \qquad 1{\cdot}2 + 2{\cdot}3 + 3{\cdot}4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

Begin by answering the following questions:

(a) State the proposition $P(n)$ to be proved and the Base Case (and prove it)

$P(n)$:= $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n\cdot(n+1) = \frac{1}{3}n(n+1)(n+2)$

Base Case (with proof):

$P(1)$: $1\cdot(1+1) = 2 = 1/3 \cdot 1 \cdot 2 \cdot 3$ ✓

(b) State the Induction Step (using variable k):

$\forall k \geq 1 \, (P(k) \rightarrow P(k+1))$

(c) State the Induction Hypothesis (I.H.) using variable k.

$P(k)$, i.e. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k\cdot(k+1) = \frac{1}{3}k(k+1)(k+2)$

(d) Now, using the above facts, give the complete proof by induction of the result above. Indicate clearly where you use the Induction Hypothesis (I.H.), as in class.

$P(1)$ – true by part (a). Suppose $P(k)$, for $k \geq 1$. We must prove $P(k+1)$ – i.e. $1\cdot 2 + 2\cdot 3 + \cdots + k\cdot(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$

Starting with LHS:

$$1\cdot 2 + 2\cdot 3 + \cdots + k(k+1) + (k+1)(k+2) \underset{\text{I.H.}}{=} \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= (k+1)(k+2)\left[\frac{1}{3}k + 1\right]$$

$$= (k+1)(k+2)\left[\frac{k+3}{3}\right]$$

$$= \frac{1}{3}(k+1)(k+2)(k+3) \quad ✓$$

So $\forall k \geq 1 \, (P(k) \rightarrow P(k+1))$. So by Induction, $\forall n \geq 1 \; P(n)$.

Additional work space. Please do not detach this page.