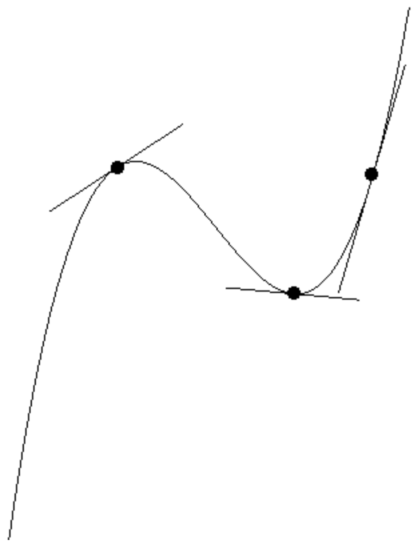
2 Differentiation

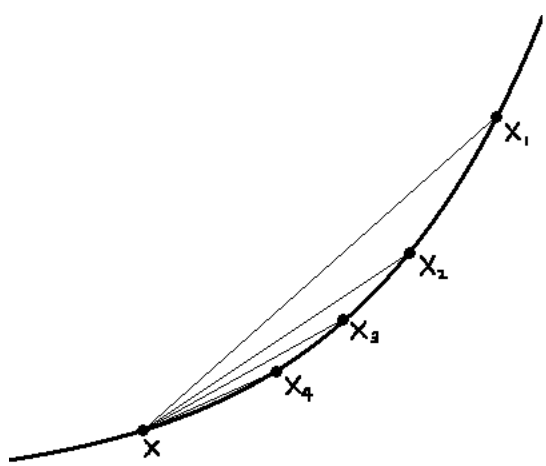
2.1 The Derivative and the Slope of Tangent Lines

When given a graph of a function, we want to be able to calculate the slope of tangent lines.



The slope of the tangent line at a point will tell us how rapidly the function is increasing or decreasing. We use a limit to find the slopes of tangent lines.

A **secant line** for a function is one that intersects it at at least two points. The idea is to approximate the tangent line using secant lines. The slope of the tangent line will be the limit of the slopes of the secants, if it exists.



The slope of the secant line connecting x_1 to x is

$$\frac{f(x_1) - f(x)}{x_1 - x}$$

If we pick a point near to x and denote it $x + \Delta x$, the slope of the secant is

$$\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We should think of Δx as being small, and note it can be positive or negative.

So, ideally, the slope of the tangent line is

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

So we make the following definition.

Definition 2.1 *Let $f(x)$ be a function. Then we define **the derivative of f at** x, denoted $f'(x)$, as*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

*if that limit exists, and if it does we say that f is **differentiable** at x.*

Notation: $f'(x), \frac{df}{dx}, y', \frac{dy}{dx}$.

Examples:

1. Find the derivative of $f(x) = \frac{1}{2}x + 3$ at $x = 2$.

   solution: We calculate:

$$\begin{aligned} f'(2) &= \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}(2+\Delta x) + 3 - (\frac{1}{2}2 + 3)}{\Delta x} \\ &= \frac{\frac{1}{2}\Delta x}{\Delta x} = \frac{1}{2} \end{aligned}$$

   So the slope of the tangent line is $m = \frac{1}{2}$. This makes sense because $y = \frac{1}{2}x + 3$ is a line, and the tangent line is the line itself.

2. Find the slope fo the tangent line for $g(x) = 1 - x^2$ at x (unspecified).

solution: We have

$$\begin{aligned} g'(x) &= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1-(x+\Delta x)^2 - (1-x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1-(x^2+2x\Delta x + \Delta x^2) - (1-x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-2x - \Delta x) \\ &= -2x \end{aligned}$$

So the slope of the tangent line at x is $-2x$.

3. Find the **equation** of the tangent line to $f(x) = \frac{1}{x}$ at the point $(1,1)$.

   solution: First we have to find the slope, then we need to use it and the point $(1,1)$ to find the line's equation.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{(-\Delta x)}{x(x+\Delta x)} \\ &= \lim_{\Delta x \to 0} \frac{-1}{x^2 + x\Delta x} \\ &= \frac{-1}{x^2} \end{aligned}$$

   So $f'(1) = -1$. Recall that $y = mx + b$, so here $y = -x + b$. We sub in the point $(1,1)$ to find b:

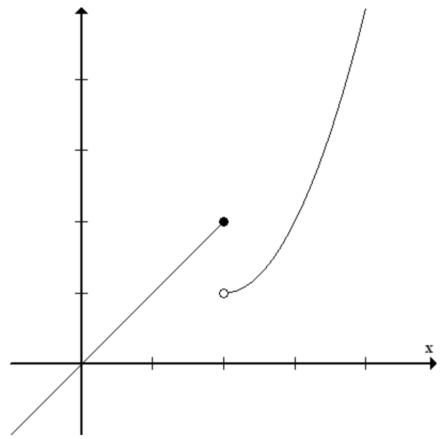
$$1 = -1 + 2 \implies b = 2 \implies y = -x + 2.$$

<u>Notes:</u>

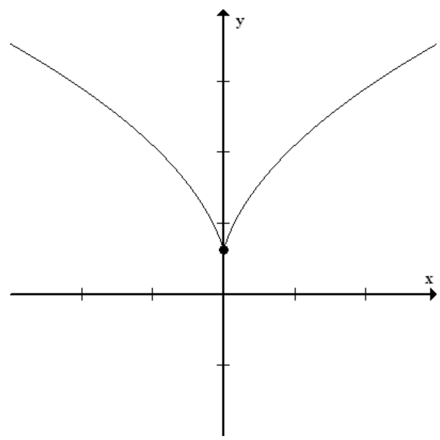
1. As you can see, calculating derivatives from the definition is difficult sometimes. We will develop rules to avoid this.

2. Not every function is differentiable everywhere. In fact, you can't take the derivative at points

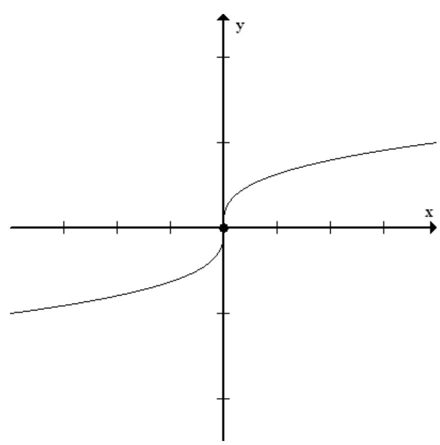
   - of discontinuity,



   - with corners,



   - or with vertical tangents.



Note that saying that a function isn't differentiable at points of discontinuity is equivalent to saying that differentiable at $x \Longrightarrow$ continuous at x.

2.2 Rules for Differentiation

We would like to avoid the limit calculation for the derivative entirely, so we state some rules for finding the derivative directly.

1. Constant Rule:
$$\frac{d}{dx}(c) = 0 \text{ for any constant } c$$
2. Power Rule: If n is any real number, $n \neq 0$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
So, for example,
$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(x^7) = 7x^6$$
3. Constant Multiple Rule:
$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$
4. Sum Rule:
$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Examples:

1. Find the derivative of $f(t) = t^{2/3} + 3t - 3$.

   solution: We use a combination of the sum rule, constant multiple rule, and power rule:
$$\begin{aligned} \frac{d}{dt}(t^{2/3} + 3t - 3) &= \frac{2}{3}t^{2/3-1} + 3t^{1-1} + 0 \\ &= \frac{2}{3}t^{-1/3} + 3 \end{aligned}$$

2. Find the derivative of $f(x) = \frac{4}{x^3}$.

   solution: Here we use the power rule and contant multiple rule:
$$\begin{aligned} \frac{d}{dx}\left(\frac{4}{x^3}\right) &= \frac{d}{dx}4x^{-3} \\ &= 4\frac{d}{dx}x^{-3} \\ &= 4(-3)x^{-3-1} \\ &= -12x^{-4} \\ &= \frac{-12}{x^4} \end{aligned}$$

2.4 Product and Quotient Rules

In addition to the rules above, we have rules for taking the derivatives of the products and quotients of functions.

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Examples

1. Differentiate $f(x) = (x^2+1)(2x+5)$ (differentiate means "find the derivative of").

   solution: There are two ways to do this. The first is to multiply the expression out and then take the derivative term-by-term. Since we have just given the product rule, let's do it that way instead.

$$\begin{aligned} f'(x) &= (x^2+1)'(2x+5) + (x^2+1)(2x+5)' \\ &= (2x)(2x+5) + (x^2+1)(2) \\ &= 6x^2 + 10x + 2 \end{aligned}$$

2. Find the derivative of $g(x) = \left(\frac{1}{x}+3\right)\left(\frac{1}{x^2}-4\right)$.

   solution: Once again we could do it two different ways, but let's use the product rule.

$$\begin{aligned} g'(x) &= \left(\frac{1}{x}+3\right)'\left(\frac{1}{x^2}-4\right) + \left(\frac{1}{x}+3\right)\left(\frac{1}{x^2}-4\right)' \\ &= \left(\frac{-1}{x^2}\right)\left(\frac{1}{x^2}-4\right) + \left(\frac{1}{x}+3\right)\left(\frac{-2}{x^3}\right) \\ &= \frac{-1}{x^4} + \frac{4}{x^2} - \frac{2}{x^4} - \frac{6}{x^3} \\ &= \frac{-3}{x^4} - \frac{6}{x^3} + \frac{4}{x^2} \end{aligned}$$

3. Find the derivative of $h(x) = \frac{\sqrt{x}}{x+1}$.

   solution: We use the quotient rule:

$$\begin{aligned} h'(x) &= \frac{(\sqrt{x})'(x+1) - (x+1)'(\sqrt{x})}{(x+1)^2} \\ &= \frac{\frac{1}{2}x^{-1/2}(x+1) - (x^{1/2})(1)}{(x+1)^2} \\ &= \frac{\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} - x^{1/2}}{(x+1)^2} \\ &= \frac{-\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}}{(x+1)^2} \end{aligned}$$

4. Let $f(x) = \frac{x^2-x-3}{x^2+1}$

   (a) Find $f'(x)$.

   (b) Find the equation of the tangent line to f at $(1, -\frac{3}{2})$.

   solution:

   a) We use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^2-x-3)'(x^2+1) - (x^2-x-3)(x^2+1)'}{(x^2+1)^2} \\ &= \frac{(2x-1)(x^2+1) - (x^2-x-3)(2x)}{(x^2+1)^2} \\ &= \frac{x^2+8x-1}{(x^2+1)^2} \end{aligned}$$

   b) The tangent line will have the form $y = mx + b$. The slope of the tangent line is the value of the derivative when $x = 1$, ie $m = f'(1) = 2$. Thus our equation is $y = 2x + b$. To find b we plug in the coordinates $(1, -\frac{3}{2})$:

$$y = 2x + b \Rightarrow -\frac{3}{2} = 2 + b \Rightarrow b = -\frac{7}{2}$$

   So the equation of the tangent line is $y = 2x - \frac{7}{2}$.

Although these rules will help when trying to take a derivative of a product or quotient, they aren't always needed. Sometimes some simplification will make the derivative easier. Here are some examples of when you don't need the rules:

1. $f(x) = \frac{1}{7}(5 - 6x^2)$. The first part is a constant, so we can just multiply through:

$$f(x) = \frac{5}{7} - \frac{6}{7}x^2 \Rightarrow f'(x) = -\frac{12}{7}x.$$

2. $g(x) = \frac{x+1}{\sqrt{x}}$. Instead of using the quotient rule, we can notice

$$g(x) = \frac{x+1}{\sqrt{x}} = \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$$

So $g'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$.

3. $h(x) = \frac{4}{5x^2} = \frac{4}{5}x^{-2}$. So $h'(x) = -\frac{8}{5}x^{-3}$.

4. $k(x) = \frac{x^2-4}{x+2} = \frac{(x+2)(x-2)}{x+2} = x - 2$ if $x \neq -2$. So $k'(x) = 1$, but both k and k' are undefined at $x = -2$.

2.3 Rates of Change: Velocity and Marginals

The derivative of $f(x)$ at c has two interpretations:

- Slope of the tangent line at c
- Rate of change of f at c.

In the second interpretations, $f'(x)$ measures the rate of change of $f(x)$ at the point $(x, f(x))$. This is the "instantaneous rate of change". one can also discuss the "average rate of change" or "rate of change over an interval":

Definition 2.2 *If $y = f(x)$, then* ***the average rate of change of y with respect to x*** *on the interval $[a, b]$ is*

$$\frac{f(b) - f(a)}{b - a}.$$

This makes sense as the average rate of change because it is the total change in y divided by the total change in x. Note that there is no limit here.

Example:

An object is dropped from a height of 100 feet. Its height at time t is given by

$$h(t) = -16t^2 + 100.$$

Find the average velocity over the following intervals:

a) $[1, 2]$

b) $[1, 1.5]$

c) $[1, 1.1]$

solution: Since $h(t)$ is a distance function, its derivative represents the rate of change of distance, ie the velocity. So we can use the average rate of change formula above.

a) We just plug the values into the formula:

$$\begin{aligned}\text{avg. vel.} &= \frac{h(b)-h(a)}{b-a}\\ &= \frac{h(2)-h(1)}{2-1}\\ &= 36-84\\ &= -48 \text{ ft/s}\end{aligned}$$

Note that the velocity is negative because the object is descending.

b)

$$\begin{aligned}\text{avg. vel.} &= \frac{h(b)-h(a)}{b-a}\\ &= \frac{h(1.5)-h(1)}{1.5-1}\\ &= \frac{64-84}{0.5}\\ &= -40 \text{ ft/s}\end{aligned}$$

c)

$$\begin{aligned}\text{avg. vel.} &= \frac{h(b)-h(a)}{b-a}\\ &= \frac{h(1.1)-h(1)}{1.1-1}\\ &= \frac{80.64-84}{0.1}\\ &= -33.6 \text{ ft/s}\end{aligned}$$

Notice that as we let the right endpoint approach 1, we get the derivative at 1 which represents the "instantaneous rate of change" at $x = 1$.

Marginal Analysis

In economics we are interested in 3 functions of the number of units sold, x:

- $P(x)$ = profit for x units
- $R(x)$ = revenue for x units
- $C(x)$ = cost for x units

These are related by the equation $P(x) = R(x) - C(x)$. The derivatives of these functions are called **marginal functions**:

- $P'(x) =$ marginal profit
- $R'(x) =$ marginal revenue
- $C'(x) =$ marginal cost

Significance? Consider the following:

Example: Suppose the cost of producing x units of a product is given by

$$C(x) = 3.6\sqrt{x} + 500$$

a) Find the additional cost when productions increases from 9 to 10 units.

solution: To find the increase in cost, we just take $C(10)$ and subtract $C(9)$:

$$C(10) - C(9) = (3.6\sqrt{10} + 500) - (3.6\sqrt{9} + 500) \approx 0.584$$

b) Find the marginal cost when $x = 9$.

solution: This amounts to finding $C'(9)$.

$$C'(x) = 3.6\left(\frac{1}{2}\right) \cdot \frac{1}{\sqrt{x}} = \frac{1.8}{\sqrt{x}}$$

Thus

$$C(9) = \frac{1.8}{3} = 0.6$$

Note that the cost of producing the 10th unit is almost exactly the marginal cost of the 9th unit.

Fact - this will generally be the case.

General Rule: for any of these functions $P(x), R(x)$, or $C(x)$, the profit, revenue or cost of producing the $(x+1)$'st unit is quite close to the marginal profit, revenue, or cost at x units.

Example: Supposse the profit for selling x units is given by $P(x) = 0.0002x^3 + 10x$.

a) Find the marginal profit at $x = 50$ units.

b) Find the gain in profit obtained by increasing production from 50 units to 51 units.

solution:

a) We have

$$\begin{aligned} P'(x) &= 0.0006x^2 + 10 \\ P'(50) &= 0.0006(50)^2 + 10 \\ &= 11.50 \end{aligned}$$

b) We must calculate $P(51) - P(50)$.

$$\begin{aligned} P(51) - P(50) &= \left[0.0002(51)^3 + 10(51)\right] - \left[0.0002(50)^3 + 10(50)\right] \\ &= 11.53 \end{aligned}$$

Again, not how close the two values are.

Demand Function

Up to this point we have dealt with some pretty idealized cost, demand, and revenue functions, some of which assumed that items could be sold at the same price for any number of units. In the real world of course sometimes the only way you will be able to sell more units is to drop the price. This leads to what is called a **demand function**.

If x is the number of units consumers are willing to purchase at a price p, then we get a function $p = D(x)$. This is the demand function.

Given a demand function $p = D(x)$, then one can recover the revenue function:

$$\boxed{R(x) = xD(x)}$$

Example: A business sells 2000 items per month at a price of \$10 each. They estimate that sales will increase by 250 units for each \$0.25 reduction in price. Find the demand and revenue functions.

solution: There are two ways of tackling this problem. I will first present the way we did it in class and then another way.

1. For every \$0.25 we drop the price we will sell 250 more units. Thus, the additional units we sell will be equal to 250 multiplied by how many quarters we drop the price by. If p is our price, then $10 - p$ represents how much we have dropped the price by. Thus, the amount of quarters we have dropped the price by is

$$\frac{10 - p}{0.25}$$

Since we were already selling 2000 units per month and we add to this the units we sell from lowering the price, the number of units we will sell, x is given by

$$x = 2000 + 250\left(\frac{10 - p}{0.25}\right)$$

Solving this for p, we end up with

$$p = -\frac{x}{1000} + 12 = D(x)$$

This is our demand function. To get revenue, we multiply by x,

$$R(x) = xD(x) = -\frac{x^2}{1000} + 12x$$

2. The number of units sold x increases by 250 every time the price p drops by \$0.25. Therefore, p and x must have a linear relationship:

p	x
10	2000
9.75	2250
9.5	2500
9.25	2750

Constant changes in x correspond to constant changes in p. Thus we can write

$$p = mx + b$$

The slope m can be calculated by any difference quotient:

$$m = \frac{\Delta p}{\Delta x} = \frac{10 - 9.75}{2000 - 2250} = \frac{-1}{1000}$$

So

$$\begin{aligned} p &= \frac{-1}{1000}x + b \\ 10 &= \frac{-1}{1000}2000 + b \\ \Rightarrow b &= 12 \\ \Rightarrow p &= \frac{-1}{1000}x + 12 \end{aligned}$$

This is the same equation we got in a).

2.5 The Chain Rule

The chain rule allows us to take the derivative of composite functions.

<u>The Rule:</u> Suppose $y = f(v)$ is a differentiable function of v and that $v = g(x)$ is a differentiable function of x. Then

- $y = f(g(x))$ is a differentiable function of x, and

- $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

  or

  $$[f(g(x))]' = f'(g(x))g(x)$$

A crucial case we will come across a lot is when $y = [g(x)]^n$. In this case,

$$y' = n[g(x)]^{n-1} \cdot g'(x)$$

This is called the **general power rule**.

Examples:

1. $h(x) = \sqrt{2x+3} = (2x+3)^{1/2}$. Then

$$h'(x) = \frac{1}{2}(2x+3)^{-1/2} \cdot 2 = (2x+3)^{-1/2}$$

2. $f(t) = (9t+2)^{2/3}$. Then

$$f'(t) = \frac{2}{3}(9t+2)^{-1/3} \cdot 9 = 6(9t+2)^{-1/3}$$

3. $g(x) = \frac{1}{\sqrt{25+x^2}} = (x^2+25)^{-1/2}$. Then

$$g'(x) = -\frac{1}{2}(x^2+25)^{-3/2} \cdot 2x = \frac{-x}{(x^2+25)^{3/2}}$$

4. Sometimes you have to combine rules. If $h(x) = x(5x+1)^3$, then we have to use the product rule and the chain rule.

$$\begin{aligned} h'(x) &= 1(5x+1)^3 + x[(5x+1)^3]' \\ &= (5x+1)^3 + x[3(5x+1)^2 \cdot 5] \\ &= (5x+1)^2(5x+1+15x) \\ &= (5x+1)^2(20x+1) \end{aligned}$$

On a test you usually won't be told which rule/rules to use. It will usually be clear which one I want you to use.

$y = (3x^2+7)(x^2-2x)$	Product Rule
$y = \sqrt[3]{x^2+1}$	Chain Rule
$y = \frac{t}{(1-t)^3}$	Quotient Rule, Chain Rule
$y = [(x-2)(x+4)]^2$	Product Rule, Chain Rule

2.6 Higher-Order Derivatives

Definition 2.3 *The* ***second order derivative****, or just* ***second derivative*** *of $f(x)$, if it exists, is*

$$f''(x) = \frac{d}{dx} f'(x)$$

The ***third derivative*** *is*

$$f'''(x) = \frac{d}{dx} f''(x)$$

and so on. For higher order derivatives we use superscripts: $f^{(4)}$ = fourth derivative etc.

I think we can all agree that something like $f''''''''''(x)$ looks rather silly - this is the reason we condense the primes after three.

An interpretation of the second derivative which frequently pops up is that of acceleration. If $f(t)$ is a distance function, ie $f(t)$ measures distance travelled at time t, then the derivative measures velocity:

$$\frac{\Delta f}{\Delta t} = \frac{\text{change in distance}}{\text{change in time}} = \text{velocity}$$

The second derivative measures acceleration

$$\frac{\Delta f'}{\Delta t} = \frac{\text{change in velocity}}{\text{change in time}} = \text{acceleration}$$

Examples:

1. Let $f(x) = x\sqrt{4-x^2}$. Find $f''(x)$.

   solution: We can use our usual rules to find the first and then second derivative:

$$\begin{aligned}
f(x) &= x\sqrt{4-x^2} \\
&= x(4-x^2)^{1/2} \\
f'(x) &= (4-x^2)^{1/2} + x\left[\frac{1}{2}(4-x^2)^{-1/2}(-2x)\right] \\
&= (4-x^2)^{1/2} - x^2(4-x^2)^{-1/2} \\
f''(x) &= \frac{1}{2}(4-x^2)^{-1/2}(-2x) - \left[2x(4-x^2)^{-1/2} + x^2\left(-\frac{1}{2}\right)(4-x^2)^{-3/2}(-2x)\right] \\
&= -x(4-x^2)^{-1/2} - 2x(4-x^2)^{-1/2} - x^3(4-x^2)^{-3/2} \\
&= -3x(4-x^2)^{-1/2} - x^3(4-x^2)^{-3/2}
\end{aligned}$$

2. An astronaut standing on the surface of the moon throws a rock into the air. The height in feet of the rock is given by

$$h(t) = -\frac{27}{10}t^2 + 27t + 6$$

Find the rock's acceleration.

solution:

$$h'(t) = -\frac{27}{5}t + 27$$
$$h''(t) = -\frac{27}{5}\ \text{ft/s}^2.$$

Just like on the earth, acceleration due to gravity is constant.

2.7 Implicit Differentiation

When we are given a function

$$y = f(x)$$

we say that y is defined **explicitly** in terms of x. In other cases, y may only be **implicitly** related to x, as is the case in the equation for the circle of radius 1:

$$x^2 + y^2 = 1$$

When we want to find $y' = \frac{dy}{dx}$ in cases like this, we must use a technique called **implicit differentiation** that relies on the chain rule.

Examples:

1. Suppose $10x + y = 2$. Find y'.

   solution: Note that in this case we could just solve for y and take the derivative, but we will do it a different way to illustrate the technique of implicit differentiation. To find y' we take the derivative of both sides;

   $$(10x + y)' = (2)'$$

   The derivative of $10x$ is 10, the derivative of y is y' and the derivative of 2 is 0, so

   $$10 + y' = 0 \ \Rightarrow y' = -10$$

2. Suppose $2x + y^2 = 2y$. Find y'.

   solution: Here again we take the derivative of both sides. When we take the derivative of the y term we use the general power rule (because y depends on x):

   $$(2x + y^2)' = (2y)'$$
   $$2 + 2(y)y' = 2y'$$

   Now we can solve for y',

   $$y' = \frac{1}{1 - y}$$

   In the above example we essentially took the derivative of y^2 normally, but then multiplied by y'. This is what will usually happen.

3. Suppose $2x + y^3 = 2y$. Find y'.

   solution: As before,

$$(2x + y^3)' = (2y)'$$
$$2 + 3(y^2)y' = 2y'$$

   Now we can solve for y',

$$y' = \frac{2}{2 - 3y^2}$$

4. Suppose $\sqrt{xy} = x - 2y$. Find y' at the point $(4, 1)$.

   solution: Here we find y' as above but then substitute in the point $(4, 1)$. Notice that the point $(4, 1)$ satisfies the equation

$$\sqrt{4 \cdot 1} = 4 - 2(1)$$

   otherwise the question would not make sense. So we take the derivative of both sides:

$$((xy)^{1/2})' = (x - 2y)'$$
$$\frac{1}{2}(xy)^{-1/2}(xy' + y) = 1 - 2y'$$
$$y'\left(\frac{x}{2}(xy)^{-1/2} + 2\right) = 1 - \frac{y}{2}(xy)^{-1/2}$$

$$\begin{aligned} y' &= \frac{1 - \frac{y}{2}(xy)^{-1/2}}{\left(\frac{x}{2}(xy)^{-1/2} + 2\right)} \\ &= \frac{1 - \frac{1}{2}(4 \cdot 1)^{-1/2}}{\left(\frac{4}{2}(4 \cdot 1)^{-1/2} + 2\right)} \\ &= \frac{1 - \frac{1}{4}}{1 + 2} \\ &= \frac{1}{4} \end{aligned}$$

4.3 Derivatives of Exponential Functions

Now that we have some tools for taking derivatives, we would like to know how to take derivatives of functions involving the exponential function. We will do this from the definition.

Recall the definition of e

$$e = \lim_{z \to 0}(1 + z)^{1/z}$$

I picked z because we are going to use x later. In fact, it doesn't matter what you put as the variable in your limit. For instance

$$e = \lim_{x \to 0}(1 + x)^{1/x} = \lim_{z \to 0}(1 + z)^{1/z} = \lim_{\clubsuit \to 0}(1 + \clubsuit)^{1/\clubsuit}$$

For reasons that will be clear in a moment, I want to use the following

$$e = \lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x}$$

Since in this limit Δx is approaching 0 through very small numbers, we can see that for very small values of Δx,

$$e \approx (1 + \Delta x)^{1/\Delta x}$$

So

$$e^{\Delta x} \approx (1 + \Delta x)$$

Now we are ready to find the derivative of e^x by using the definition as a limit. Let $f(x) = e^x$. Then

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x e^\Delta x - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x (e^{\Delta x} - 1)}{\Delta x} \\
&\approx \lim_{\Delta x \to 0} \frac{(1 + \Delta x - 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x \Delta x}{\Delta x} \\
&= e^x
\end{aligned}$$

So from this it appears that $f'(x) = e^x$, and that is indeed the case. Thus e^x is its own derivative. Further,

$$\frac{d}{dx} e^x = e^x \quad \frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x)$$

The last is of course the chain rule.

<u>Examples:</u>

1. $$y = e^{7x} \Rightarrow y' = 7e^{7x}$$

2. $$y = 5e^{x^4} \Rightarrow y' = 5e^{x^4}(4x^3) = 20x^3 e^{x^4}$$

3.

$$\begin{aligned} y &= e^{\sqrt{2x+1}} \\ y' &= e^{\sqrt{2x+1}}(\sqrt{2x+1})' \\ &= e^{\sqrt{2x+1}}\frac{2}{2\sqrt{2x+1}} \\ &= \frac{e^{\sqrt{2x+1}}}{\sqrt{2x+1}} \end{aligned}$$

4.

$$y = x^2 e^x \ \Rightarrow y' = 2xe^x + x^2 e^x$$

5. Find the equation of the tangent line for the function $y = (e^{-x} + e^{2x})^3$ when $x = 0$.

   <u>*solution:*</u> First note that when $x = 0$, $y = (1+1)^3 = 8$, so the line goes through $(0, 8)$ and therefore its y-intercept is 8.

$$y' = 3(e^{-x} + e^{2x})^2(-e^{-x} + 2e^{2x})$$

$$y'(0) = 3(2)^2(-1+2) = 12$$

   The slope of the line is 12, and above we found the y-intercept to be 8. Thus the equation we need is $y = 12x + 8$.

6. Given $e^{xy} + x^2 + y^2 = -8$, find $\frac{dy}{dx}$ at $(0, 3)$.

   <u>*solution:*</u> Since y is implicitly related to x we have to use implicit differentiation:

$$e^{xy}(xy' + y) + 2x - 2yy' = 0$$

   Plugging in the point $(0, 3)$ gives us

$$1(0 \cdot y' + 3) + 2(0) - 6y' = 0$$

$$3 - 6y' = 0 \Rightarrow y' = \frac{1}{2}$$

We will mostly be interested in $y = e^x$, but we can consider other bases.

Theorem 4.1 *Let $b > 0, b \neq 1$. If $y = b^x$, then $y' = b^x ln(b)$.*

<u>proof</u>: We can rewrite b^x:

$$b^x = \left[e^{\ln(b)}\right]^x = e^{x\ln(b)}$$

So by the chain rule

$$\begin{aligned} y' &= e^{x\ln(b)} \cdot \ln(b) \\ &= \left[e^{\ln(b)}\right]^x \ln(b) \\ &= b^x \ln(b) \end{aligned}$$

This completes the proof. ■

Examples:

1. $$y = 3^{x^5} \quad \Rightarrow y' = \ln(3)3^{x^5}(5x^4)$$

2. $$\begin{aligned} y &= \frac{5^{4x}}{\sqrt{x}} \\ y' &= \frac{\ln(5)(5^{4x})4\sqrt{x} - 5^{4x}\left(\frac{1}{2}x^{-1/2}\right)}{x} \\ &= \frac{5^{4x}(8\ln(5)x - 1)}{2x^{3/2}} \end{aligned}$$

4.5 Derivatives of Logarithms

Since we know the derivative of exponential functions, it's natural to want to know the derivative of logarithms. We can find these derivatives easily using implicit differentiation.

Suppose $y = \ln(x)$. Then by the rules of logs, $x = e^y$. Using implicit differentiation on this gives us

$$\begin{aligned} 1 &= e^y \cdot y' \\ y' &= \frac{1}{e^y} \end{aligned}$$

But $e^y = x$, so $y' = \frac{1}{x}$. This deserves a box:

$$\boxed{\text{If } f(x) = \ln(x), \text{ then } f'(x) = \tfrac{1}{x}}$$

Examples:

1. Let $y = x^2 \ln(x)$. Then
$$y' = 2x\ln(x) + x^2\frac{1}{x} = x + 2x\ln(x)$$

2. Let $y = \ln(3x^2 + 7x)$. Then
$$y' = \frac{1}{3x^2 + 7x}(6x + 7)$$

3. Let $y = \ln(\sqrt{x^3 + 7x})$. Then we can take the derivative directly or we can notice first that we can simplify this with the laws of logs:
$$y = \ln(\sqrt{x^3 + 7x}) = \ln((x^3 + 7x)^{1/2}) = \frac{1}{2}\ln(x^3 + 7x)$$

So that

$$y' = \frac{1}{2}\frac{1}{x^3+7x}(3x^2+7) = \frac{3x^2+7}{2x^3+14x}$$

4. Suppose $y = \frac{\ln(x)}{x}$. Find the equation of the tangent line at $x = e$.

   solution: Note that when $x = e$, $y = \frac{1}{e}$. We take the derivative so that we can find the slope:

$$\begin{aligned} y' &= \frac{\left(\frac{1}{x}\right)x - \ln(x)(1)}{x^2} \\ &= \frac{1-\ln(x)}{x^2} \end{aligned}$$

   Plugging in $x = e$ gives us $\frac{1-\ln(e)}{e^2} = 0$. So the slope is 0, and our equation so far is $y = 0x + b$. Plugging in the point $\left(e, \frac{1}{e}\right)$ gives us $b = \frac{1}{e}$, so the equation of the tangent line is $y = \frac{1}{e}$.

4.6 Exponential Growth and Decay

Many naturally occuring systems involve a quantity whose rate of change is proportional to how much of that quantity is present. Bacterial growth is a perfect example - the more bacteria there are in a colony, the faster the colony grows due to cell division. Cases like this are modelled by the equation

$$\frac{dy}{dt} = ky$$

That is, the rate of change of y is proportional to y (here k is a fixed constant). In this case, it can be shown that y has the form

$$y = Ce^{kt}$$

Where C is the **initial value** and k is the **constant of proportionality**. If $k > 0$, we say that y **grows exponentially** and if $k < 0$ we say that y **decays exponentially**.

Note that the reason we say C is the initial value is that when $t = 0$, $y = Ce^0 = C$.

Also note that in other places (specifically some of the old midterms) you will see $y = Cb^t$ for some b instead. The two are equivalent if you take $b = e^k$.

Examples:

1. Our formula for continuous compounding

$$A(t) = Pe^{rt}$$

   is a typical example of exponential growth - at all times the rate of change of $A(t)$ is proportional to the interest rate:

$$\frac{dA}{dt} = rA$$

2. The number of bacteria in a certain culture is 160,000 at 1pm. At 3pm, the count is 320,000. Assuming the population is growing exponentially, find the population at 7pm.

<u>*solution:*</u> Since we are assuming exponential growth, we use the equation

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial population. Taking 1pm to be $t = 0$, $P_0 = 160{,}000$. To be able to use this formula we need to find k, and to do this we use the fact that at $t = 2$, $P(t) = 320{,}000$:

$$\begin{aligned} P(t) &= 160000e^{kt} \\ P(2) &= 160000e^{2k} = 320000 \\ e^{2k} &= 2 \\ 2k &= \ln(2) \\ k &= \frac{\ln(2)}{2} \end{aligned}$$

So $P(t) = 160000e^{\frac{\ln(2)}{2}t}$. At 7pm, $t = 6$, so

$$P(6) = 160000e^{3\ln(2)} = 1{,}280{,}000$$

3. Radioactive substances decay exponentially, meaning their mass decreases in an exponential fashion. The **half-life** of a radioactive substance is the time it takes for its size to be reduced to half of the initial amount.

The half-life of a certain radioactive substance is 25 years. If 20 grams were present in 1985, how much will be left in 2025?

<u>*solution:*</u> We again assume that $P(t) = P_0 e^{kt}$. A half-life of 25 years means that when $t = 25$, $P(t)$ will be half of the initial amount, $\frac{P_0}{2}$:

$$\begin{aligned} \frac{P_0}{2} &= P(25) \\ &= P_0 e^{25k} \\ \frac{1}{2} &= e^{25k} \\ \ln\left(\frac{1}{2}\right) &= 25k \\ k &= \frac{\ln(\frac{1}{2})}{25} \\ &= -\frac{\ln(2)}{25} \end{aligned}$$

If we take 1985 as $t = 0$, then $P_0 = 20$. Thus

$$P(t) = 20e^{-\frac{\ln(2)}{25}t}$$

We want $P(40) = 20e^{-\frac{\ln(2)}{25}40} \approx 6.6$ grams.

4. The number of rabbits on an island grows exponentially and takes 1 year to double. How long does it take to triple?

   solution: We again assume that $P(t) = P_0 e^{kt}$. We are given that at $t = 1$, $P(t)$ will be double the initial population, ie $2P_0$. Thus

$$\begin{aligned} 2P_0 &= P_0 e^k \\ 2 &= e^k \\ k &= \ln(2) \end{aligned}$$

   Thus $P(t) = P_0 e^{\ln(2)t}$. We want t such that $P(t) = 3P_0$. Thus we solve

$$3P_0 = P_0 e^{\ln(2)t}$$

$$3 = e^{\ln(2)t}$$

$$\ln(3) = \ln(2)t$$

$$t = \frac{\ln(3)}{\ln(2)} \approx 1.58y$$