Solution to the Final Examination
MAT1322-3X, Summer 2014

Part I. Multiple-choice Questions ($3 \times 10 = 30$ marks)

1. The area of the region under the graph of $y = \frac{10x}{1+x^2}$, and above the line $y = x$ is

(A) $5 \ln 10 + \frac{9}{2}$; (B) $5 \ln 10 - \frac{9}{2}$; (C) $10 \ln 3 + \frac{9}{2}$;

(D) $10 \ln 3 - \frac{9}{2}$; (E) $3 \ln 10 + \frac{9}{2}$; (F) $5 \ln 3 - \frac{9}{2}$.

Solution. (B) Let $\frac{10x}{1+x^2} = x$. Then $x = 0$ or $x = 3$. The area is

$$\int_0^3 \left(\frac{10x}{1+x^2} - x\right)dx = \int_0^3 \frac{10x}{1+x^2}\,dx - \int_0^3 x\,dx = 5\int_1^{10} \frac{1}{u}\,du - \frac{9}{2} = 5\ln 10 - \frac{9}{2}.$$

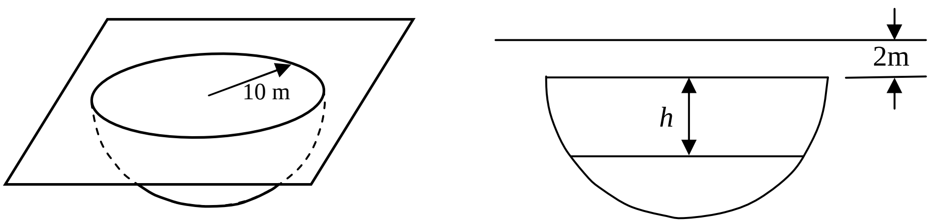
2. Let R be the region in x-y plane above the parabola $y = x^2$ and under the line $y = 2x$. Let S be a solid obtained by revolving region R about the y-axis. Then the volume of S is

(A) $\frac{1}{2}\pi$; (B) $\frac{4}{3}\pi$; (C) $\frac{3}{8}\pi$; (D) $\frac{12}{5}\pi$; (E) $\frac{14}{9}\pi$; (F) $\frac{8}{3}\pi$.

Solution. (F) For a given value of y, $r_{\text{inner}} = \frac{y}{2}$ and $r_{\text{outer}} = \sqrt{y}$. Let $\frac{y}{2} = \sqrt{y}$. $y^2 = 4y$, $y = 0, 4$.

The volume of S is $V = \pi\int_0^4 \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right)dy = \frac{8}{3}\pi$. Note that the integral is with respect to y, which takes the $y = 0$ to 4.

3. A semi-spherical tank with radius 10 meters as shown in the following figure is filled with water of density ρ kg / m^3.



Let h be the depth of a layer of water in the tank. Let g be the acceleration of gravity. The work, in Joules, needed to pump the water to a height 2 meter above the top of the tank is calculated by

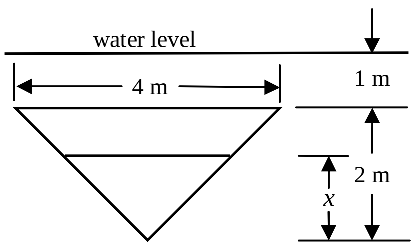
(A) $\pi\rho g\int_0^{10}(100-h^2)(h+2)dh$; (B) $\pi\rho g\int_2^{12}(100-h^2)(h+2)dh$;

(C) $\pi\rho g\int_0^{10}(10-h)^2(h+2)dh$; (D) $\pi\rho g\int_0^{12}(10-h)^2(h+2)dh$;

(E) $\pi\rho g\int_0^{10}(100-h^2)(12-h)dh$; (F) $\pi\rho g\int_2^{12}(100-h^2)(12-h)dh$.

Solution. (A) At a layer of depth h and thickness dh, the radius is $r = \sqrt{10^2-h^2}$, and the volume is $dV = \pi r^2 dh = \pi(100 - h^2)dh$. The weight is $dw = \pi\rho g(100 - h^2)dh$. To pump the water of this layer to 2 meters above the top of the tank, the work needed is $dW = (h + 2)dw$ $= \pi\rho g(100 - h^2)(h + 2)\, dh$. Hence, the total work needed is $W = \pi\rho g\int_0^{10}(100-h^2)(h+2)dh$.

4. A triangular surface is submerged into water with density ρ kg/m^3 such that the top is 1 meter deep in the water as shown in the following figure.



Let x be the distance between a horizontal layer of the surface and the bottom of the triangle. Denote the acceleration of gravity by g. Then the integral used to find the total force acting on this surface is

(A) $\int_0^3 2\rho gx(3-x)dx$; (B) $\int_0^2 2\rho gx(x+1)dx$; (C) $\int_0^2 2\rho gx(3-x)dx$;

(D) $\int_0^3 \frac{\rho gx(3-x)}{2}dx$; (E) $\int_0^2 \frac{\rho gx(3-x)}{2}dx$; (F) $\int_0^3 2\rho gx(x+1)dx$.

Solution. (C) A slice of the surface x meters from the bottom of the triangle with height dx has area $2xdx$. The depth of this slice is $D = 3 - x$. The pressure on this slice is $P = \rho gD = \rho g(3 - x)$. The force acting on this slice is $dF = PA = 2\rho gx(3 - x)dx$. The total force acting on this slice is $F = \int_0^2 2\rho gx(3-x)dx$.

5. Suppose the population of a culture of bacteria grows exponentially. At time $t = 0$, the population is 10,000, and the population reaches 30,000 at $t = 2$. At time T the population is 100,000. Then $T =$

(A) $\dfrac{\ln 10}{\ln 3}$; (B) $\dfrac{10\ln 3}{\ln 10}$; (C) $\dfrac{7\ln 10}{\ln 3+\ln 10}$; (D) $\dfrac{2\ln 10}{\ln 3}$; (E) $\dfrac{13}{\ln 3+\ln 10}$; (F) ln3 + ln10.

Solution. (D) The model is $P(t) = 10000e^{kt}$. Since $P(2) = 10000e^{2k} = 30000$, $e^{2k} = 3$, $k = \ln 3 / 2$. $1000000 = 10000e^{kT}$, $e^{kT} = 10$. $T = \ln 10 / k = 2 \ln 10 / \ln 3$.

6. Suppose Euler's method with step size $h = 0.1$ is used to estimate $y(0.3)$, where $y(t)$ is the solution to the initial-value problem $y' = 2ty^2 - y^2$, $y(0) = -1$. Which one of the following values is closest to the result that you obtained?

(A) −1.266; (B) −1.283; (C) −1.312; (D) −1.357; (E) −1.402; (F) −1.431.

Solution. (B) The iteration formula is $y_{n+1} = y_n + h(2t_n y_n^2 - y_n^2)$ with $t_0 = 0$, and $y_0 = -1$.

n	t_n	y_n
0	0	-1
1	0.1	$-1 + 0.1 \times (2 \times 0^2 \times 1 - 1^2) = -1.1$
2	0.2	$-1.1 + 0.1 \times (2 \times 0.1 \times (-1.1)^2 - (-1.1)^2) = -1.1968$
3	0.3	$-1.1968 + 0.1 \times (2 \times 0.2 \times (-1.1968)^2 - (-1.1968)^2) = -1.2827$

$y(0.3) \approx -1.2827$.

7. The sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + (-1)^n 7^n}{11^{n+1}}$ is

(A) $\dfrac{11}{18}$; (B) $\dfrac{33}{18}$; (C) $\dfrac{3}{11}$; (D) $\dfrac{18}{11}$; (E) $\dfrac{3}{18}$; (F) $\dfrac{11}{3}$.

Solution. (E) This series is the sum of two geometric series:

$$\sum_{n=0}^{\infty} \frac{2^n + (-1)^n 7^n}{11^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{11^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n 7^n}{11^{n+1}} = \frac{\frac{1}{11}}{1-\frac{2}{11}} + \frac{\frac{1}{11}}{1+\frac{7}{11}} = \frac{1}{9} + \frac{1}{18} = \frac{3}{18}.$$

8. Consider power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^n\sqrt{n}}$. Which one of the following statement is true?

(A) When $-3 < x < 3$, this series is absolutely convergent. When $x \le -3$ or $x \ge 3$, it is divergent.

(B) When $-3 < x < 3$, this series is absolutely convergent. When $x < -3$ or $x \ge 3$, it is divergent. When $x = -3$, this series is convergent but not absolutely convergent.

(C) When $-3 < x < 3$, this series is absolutely convergent. When $x \le -3$ or $x > 3$, it is divergent. When $x = 3$, this series is convergent but not absolutely convergent.

(D) When $-3 < x < 3$, this series is convergent but not absolutely convergent. When $x \le -3$ or $x \ge 3$, it is divergent.

(E) When $-3 < x < 3$, this series is convergent but not absolutely convergent. When $x < -3$ or $x \ge 3$, it is divergent. When $x = -3$, it is absolutely convergent.

(F) When $-3 < x < 3$, this series is convergent but not absolutely convergent. When $x \le -3$ or $x > 3$, it is divergent. When $x = 3$, it is absolutely convergent.

Solution. (C) Use the ratio test.

Let $\lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{3^{n+1}\sqrt{n+1}}\dfrac{3^n\sqrt{n}}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{3}\sqrt{\dfrac{n}{n+1}}\right| = \left|\dfrac{x}{3}\right| < 1$. Then $|x| < 3$, or $-3 < x < 3$.

The radius of convergence is $R = 3$. Hence, this series is absolutely convergent when $-3 < x < 3$, and it is divergent when $x < -3$ or $x > 3$.

When $x = 3$, this series becomes $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{\sqrt{n}}$, which is convergent by the alternating series test. Hence, this series convergent but not absolutely convergent at $x = 3$.

When $x = -3$, this series becomes $\sum\limits_{n=0}^{\infty}\dfrac{1}{\sqrt{n}}$, which is divergent by p-series test. Hence, this series is divergent when $x = -3$.

9. If a two variable function $z = f(x, y)$ is defined implicitly by the equation $x^2z^2 + 3xy + yz^3 - 3 = 0$, then the partial derivative $\dfrac{\partial z}{\partial x}$ at the point $(2, -1, 3)$ is

(A) 13; (B) $\dfrac{1}{13}$; (C) $-\dfrac{1}{13}$; (D) 11; (E) $\dfrac{1}{11}$; (F) -11.

Solution. (D) Let $F(x, y, z) = x^2z^2 + 3xy + yz^3 - 3$. $F_x = 2xz^2 + 3y$, $F_z = 2x^2z + 3yz^2$. $F_x(2, -1, 3) = 33$, $F_z(2, -1, 3) = -3$. $\dfrac{\partial z}{\partial x}(2,-1,3) = -\dfrac{F_x(2,-1,3)}{F_z(2,-1,3)} = -\dfrac{33}{-3} = 11$.

10. The directional derivative of the function $z = xy^2 - 3x^2 - 2x - 2y$ at the point $(2, 3)$ in the direction of vector $\mathbf{u} = \left(\dfrac{3}{5}, \dfrac{4}{5}\right)$ is

(A) 5; (B) 3; (C) 2; (D) −2; (E) −4; (F) −1.

Solution. (A) $z_x = y^2 - 6x - 2$, $z_y = 2xy - 2$. At point (2, 3), $z_x(2, 3) = -5$, $z_y(2, 3) = 10$. The directional derivative is $z_{\mathbf{u}}(2, 3) = (-5)\times\frac{3}{5}+10\times\frac{4}{5}=5$.

Part II. Long Answer Questions (20 marks)

1. (4 marks) (a) Use the definition to find the value of improper integral $\int_e^\infty \frac{1}{x(\ln x)^3}dx$.

(b) Use comparison test to show that improper integral $\int_0^1 \frac{1+x}{\sqrt{2x^3-x^4}}dx$ is divergent.

Solution. (a) $\int_e^\infty \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^3}dx$.

Use a variable substitution, $u = \ln x$:

$$\int_e^b \frac{1}{x(\ln x)^3}dx = \int_1^{\ln b} u^{-3}du = \left[-\frac{1}{2}u^{-2}\right]_{u=1}^{\ln b} = \frac{1}{2}\left(1-\frac{1}{(\ln b)^2}\right).$$

Since $\lim_{b\to\infty}\left(1-\frac{1}{(\ln b)^2}\right)=1$, this improper integral converges, and $\int_e^\infty \frac{1}{x(\ln x)^3}dx = \frac{1}{2}$.

(b) Use comparison test.

Since $2x^3 - x^4 < 2x^3$ and $x + 1 > 1$ when $0 < x < 1$, $\frac{1+x}{\sqrt{2x^3-x^4}} > \frac{1}{\sqrt{2x^3}}$. Since improper integral $\int_0^1 \frac{1}{\sqrt{2x^3}}dx = \frac{1}{\sqrt{2}}\int_0^1 x^{-3/2}dx$ is divergent, this improper integral is divergent.

2. (5 marks) Consider differential equation $\frac{dy}{dt} = y^2 - 3y - 4$.

(a) (1 mark) Find equilibrium solutions of this equation.

(b) (4 marks) Solve this equation with initial condition $y(0) = 3$.

Solution. (a) Let $y^2 - 3y - 4 = 0$. The equilibrium solutions are $y = -1$, and $y = 4$.

(b) Separating the variables: $\int \frac{1}{(y+1)(y-4)}dy = \int dt$. Use partial fraction to integrate the left-hand side:

Let $\frac{1}{(y+1)(y-4)} = \frac{A}{y+1} + \frac{B}{y-4} = \frac{A(y-4)+B(y+1)}{(y+1)(y-4)}$. Then $A(y-4) + B(y+1) = 1$. Let $y = -1$. We have $A = -\frac{1}{5}$. Let $y = 4$. We have $B = \frac{1}{5}$.

$$\int \frac{1}{(y+1)(y-4)}dy = \frac{1}{5}\left(\int \frac{1}{y-4}dy - \int \frac{1}{y+1}dy\right) = \frac{1}{5}\ln\left|\frac{y-4}{y+1}\right| + C.$$

$\frac{y-4}{y+1} = Ke^{5t}$, where K is an arbitrary constant. By the initial condition, $K = -\frac{1}{4}$.

Therefore, $16 - 4y = ye^{5t} + e^{5t}$. $y = \frac{16-e^{5t}}{4+e^{5t}}$.

3. (6 marks) Use an appropriate test method to determine whether each of the following series is convergent or divergent. State the conditions why the test method applies.

(a) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2n^2-1}$;

(b) $\sum_{n=1}^{\infty}\frac{n}{e^n}$;

(c) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\sqrt{\ln n}}$.

Solution. (a) Since the series is a positive series, we can use the limit comparison test. Let $a_n = \frac{\sqrt{n}}{2n^2-1}$, and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{2n^2-1}(n\sqrt{n}) = \lim_{n\to\infty}\frac{n^2}{2n^2-1} = \frac{1}{2}$. Since $\sum_{n=1}^{\infty}b_n$, as a p-series with $p = 3/2 > 1$, is convergent. This series is convergent.

(b) Since the general term is positive, decreasing and continuous, we can use the integral test.

$\int_0^{\infty} xe^{-x}dx = \lim_{b\to\infty}\left[-(1+x)e^{-x}\right]_{x=0}^{b} = \lim_{b\to\infty}(1-(b+1)e^{-b}) = 1 < \infty$, this series is convergent.

Alternative solution: Since the series is positive, we may also use the ratio test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)e^n}{ne^{n+1}} = \lim_{n\to\infty}\frac{1}{e}$ < 1. By the ratio test, this series is convergent.

(c) Since this series is alternating with decreasing general terms, according to alternating series test, it is convergent.

4. (5 marks) Recall that the Maclaurin series of the sine function is

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots\ .$$

(a) (2 mark) Find the Maclaurin series of the function $y = \cos(x^2)$.

(b) (3 marks) Find the Maclaurin series of a function $y = F(x)$ defined by definite integral $F(x) = \int_0^x \cos(t^2)\,dt$.

Solution. (a)

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!} = 1-\frac{x^4}{2!}+\frac{x^8}{4!}-\ldots\ .$$

(b) $\int_0^x \cos(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n\frac{t^{4n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\int_0^x\frac{t^{4n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!}$

$$= x-\frac{x^5}{5\times 2!}-\frac{x^9}{9\times 4!}+\frac{x^{13}}{13\times 6!}-\ldots\ .$$