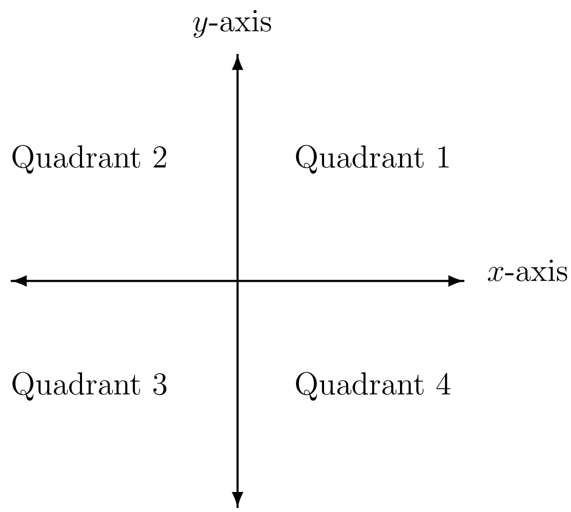
1 Functions, Graphs and Limits

1.1 The Cartesian Plane

In this course we will be dealing a lot with the *Cartesian plane* (also called the *xy-plane*), so this section should serve as a review of it and its properties.



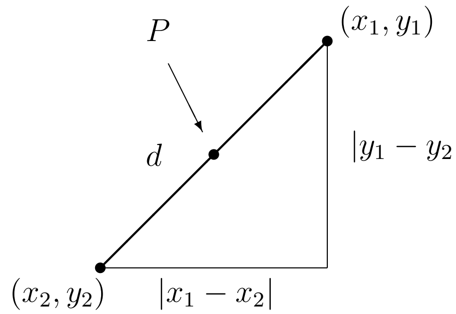
The xy-plane is divided into 4 quadrants by the x-axis and y-axis. Each point in the xy-plane is represented by an ordered pair (x, y). If we have two points (x_1, y_1), (x_2, y_2) in the xy-plane, then:

- the distance d between them is found using the Pythagorean theorem. The formula is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- the midpoint between them is found by averaging the x-coordinates and then averaging the y-coordinates. The formula is:

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



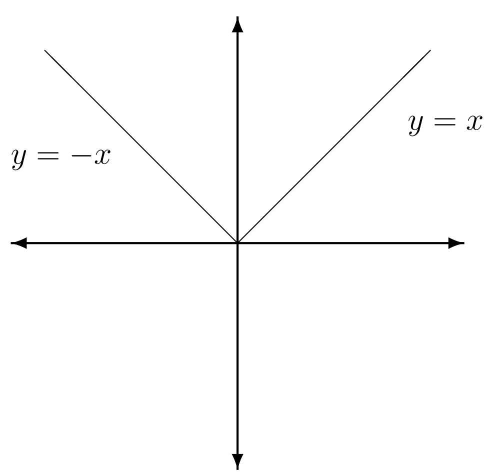
1.2 Graphs of Functions and Equations

In this section we review the graphing of functions.

Example: Graph the function

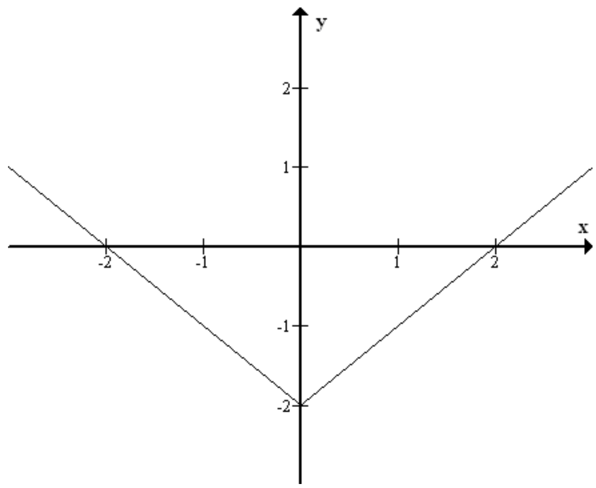
$$y = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x \leq 0 \end{cases}$$

solution:



Example: Graph the function $y = |x| - 2$.

solution: The effect of the -2 is to shift the graph down 2 units:



Terminology:

- x-intercepts - place or places where the graph crosses the x-axis. There could be several or none. To find the x-intercept for $y = f(x)$, solve $f(x) = 0$.
- y-intercept - where graph crosses the y-axis. To find, set $x = 0$. Note that the graph of a function can have at most one y-intercept.

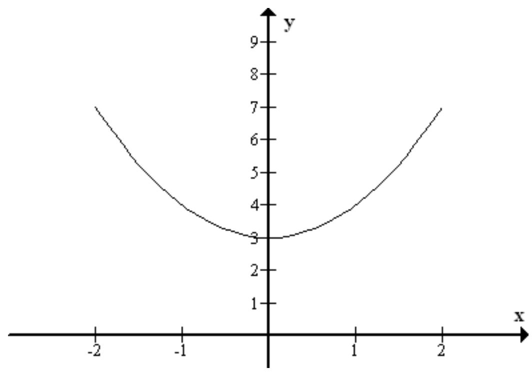
You should be familar with graphing:

- straight lines
- parabolics
- absolute values

Examples:

1. Graph and find the intercepts of the function $y = f(x) = x^2 + 3$

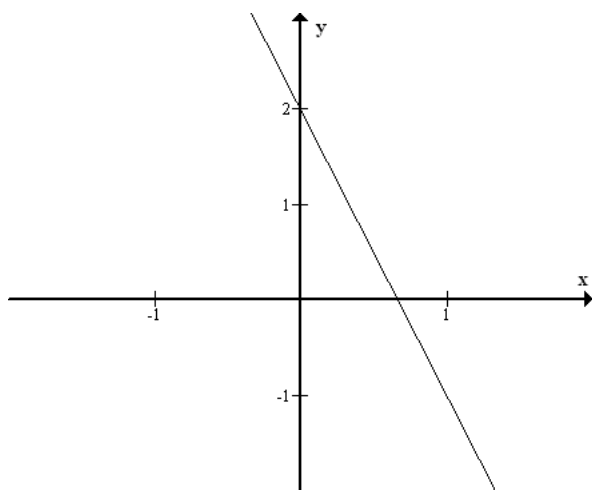
solution:



- y-intercept at $(0, 3)$
- x-intercept - none. Why?

2. Graph and find the intercepts of the function $y = f(x) = -3x + 2$

solution: This is a line with slope equal to -3.



- y-intercept - $x = 0 \Rightarrow y = 2$
- x-intercept - $y = 0 \Rightarrow x = \frac{2}{3}$

Application: Break-even Point

The breakeven point is where revenue = cost. Typically, when a business starts, its costs far exceed revenue. As additional units are produced (assuming it is a reasonable company), the business will evenually be at a point where total revenue = total cost.

Usually, after that, total revenue will exceed total cost. So the company is making a profit.

Example: Break-even analysis

You are starting a part-time business. You make an initial investment of \$6000. The cost of producing each unit is \$6.50. They are sold for \$13.90.

a) Find equations for $C(x)$, the total cost and $R(x)$, the total revenue, where x is the number of units.

b) Find the break-even point.

Note that this model is extremely oversimplified. As the course progresses, we will see more realistic examples.

solution:

a) $C(x) = \underbrace{6000}_{\text{initial cost}} + \underbrace{6.50x}_{\text{cost for } x \text{ units}}$

$R(x) = 13.90x$

b) To find the breakeven point, set $C(x) = R(x)$ and solve for x.

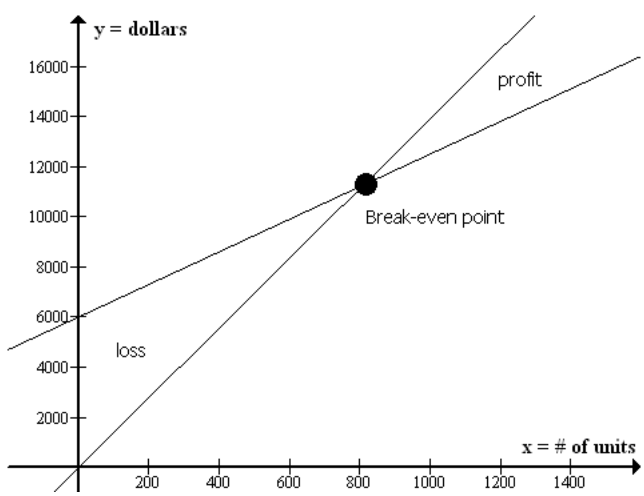
$$C(x) = R(x)$$

$$6000 + 6.50x = 13.90x$$

$$6000 = 7.4x$$

$$x = \frac{6000}{7.4} \approx 811$$

So the company must sell approximately 811 units before it breaks even. Graphically:



1.3 Lines in the plane

Recall the slope-intercept form of the equation of a line.

$y = mx + b$ where m is the slope of the line and $b$is the y-intercept of the line.

To find the slope using two points (x_1, y_1) and (x_2, y_2), use

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{“change in } y\text{”}}{\text{“change in } x\text{”}}$$

(assuming $x_1 \neq x_2$)

Examples:

1. Find an equation of the line passing through $(-3, 6)$ and $(1, 2)$.

   solution: First, get slope:

   $$m = \frac{\Delta y}{\Delta x} = \frac{2-6}{1-(-3)} = \frac{-4}{4} = -1$$

   Thus the line has equation $y = -1x + b$. Now we can plug either of the points into this equation and solve for b. To substitute the point $(-3, 6)$ into the equation, we let $x = -3$ and $y = 6$:

   $$6 = (-1)(-3) + b \ \Rightarrow b = 3$$

   So $y = -x + 3$.

2. Find an equation for the line with slope $\frac{1}{6}$ through the point $(1, 5)$.

   solution: They give us the slope, so we need to only find b. We plug the point $(1, 5)$ into $y = \frac{1}{6}x + b$ and solve:

   $$5 = \frac{1}{6}(1) + b \ \Rightarrow b = \frac{29}{6}$$

   Therefore an equation of the line is $y = \frac{1}{6}x + \frac{29}{6}$.

Exercise: Notice that in the above two examples we asked for “an” equation of the line in question – this implies that there are others. Can you think of any more?

Example: Suppose you are a contractor, and have purchased a piece of equipment for \$26,500. The equipment costs \$5.25 per hour for fuel and maintenance. The operator of the equipment is paid \$9.50 per hour.

a) Write a linear equation (that is, an equation for a line) giving the total cost $C(t)$ of operating equipment for t hours.

b) You charge your customers \$25/hr. Find an equation for revenue $R(t)$.

c) Find an equation for profit.

d) Find the number of hours you must operate before breaking even.

solution: Let t denote time in hours.

a) $C(t) = 26,500 + 5.25t + 9.50t = 26,500 + 14.75t$.

b) $R(t) = 25t$.

c) Profit = Revenue - Cost

$$P(t) = R(t) - C(t) = 25t - 26,500 - 14.75t = 10.25t - 26,500$$

d) To find the break-even point, we set $R(t) = C(t)$. Notice that this is the same as setting $P(t) = 0$.

$$25t = 26,500 + 14.75t$$

$$10.25t = 26,500$$

$$t \approx 2585.4 \text{ hours.}$$

1.4 Functions

In the expression

$$y = f(x) = x^2 + 3$$

we say that x is the **independent variable** and that y is the **dependent variable**.

A **function** is a relationship between two variables such that to each value of the independent variable there corresponds **exactly** one value of the dependent variable. This is a rewording of what you may know as the so-called "vertical line test" for functions.

The **domain** of a function is the set of all values of the independent variable for which the function is defined.

Examples:

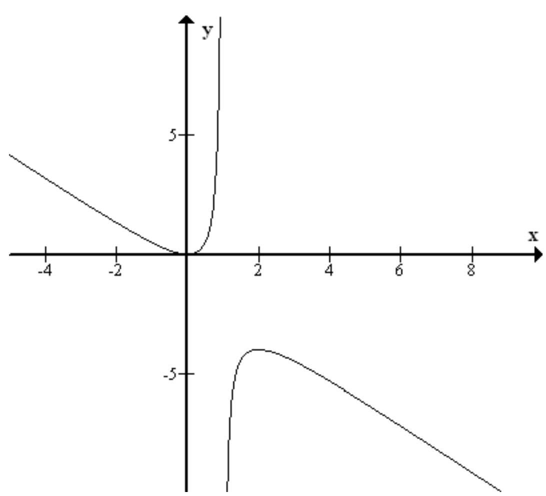
1. Find the domain of

$$f(x) = \frac{x^2}{1 - x}$$

solution: In a question like this, we are trying to find the largest possible domain. Here, the only potential problem is that the denominator could be 0. So the domain is every real number *except* 1. We write

$$\text{Domain } = \{x \mid x \neq 1\}$$

We will see later that the graph looks like this:



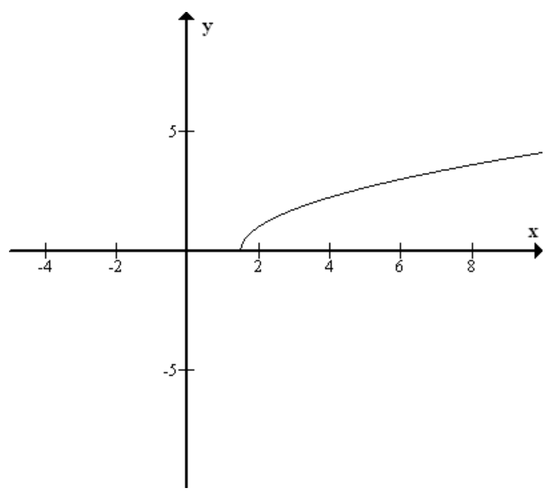
2. Find the domain of

$$f(x) = \sqrt{2x-3}$$

<u>*solution:*</u> Here we need to make sure that what's inside the square root is non-negative. So,

$$2x - 3 \geq 0$$
$$x \geq \frac{3}{2}$$

So the domain is $[\frac{3}{2}, \infty)$. Note that $\frac{3}{2}$ is included. The graph looks like:



Related to the domain is the **range**, which is the set of all y-values a function could possibly take.

<u>Composites</u>

If f and g are functions, then their composite, denoted $f \circ g$, is defined to be $f(g(x))$. In other words, apply g to x and then apply f to the result.

<u>Example:</u> If $f(x) = 1 + x^2$ and $g(x) = 2x - 1$, then

$$\begin{aligned} f(g(x)) &= f(2x-1) = 1 + (2x-1)^2 \\ &= 1 + (4x^2 - 4x + 1) \\ &= 4x^2 - 4x + 2 \end{aligned}$$

On the other hand,

$$\begin{aligned} g(f(x)) &= g(1+x^2) = 2(1+x^2) - 1 \\ &= 2x^2 + 1 \end{aligned}$$

So $f \circ g \neq g \circ f$, in general.

Inverses

Two functions f and g are **inverses** of each other if $f(g(x)) = x$ and $g(f(x) = x$. Normally, g is denoted f^{-1}. So,

$$f \circ f^{-1}(x) = x$$
$$f^{-1} \circ f(x) = x$$

Example: If $f(x) = 6 - 3x$, then I claim that $f^{-1}(x) = -\frac{1}{3}x + 2$.

Check:

$$f(f^{-1}(x)) = f(-\frac{1}{3}x + 2) = 6 - 3(-\frac{1}{3}x + 2) = 6 + x - 6 = x$$

You must also check that $f^{-1}(f(x)) = x$ as well (exercise!)

Some functions have inverses and some don't. A function is said to be **one-to-one** if no two x-values get sent to the same y-value. If a function is one-to-one, then it passes the so-called "horizontal line test": any horizontal line in the xy-plane will only intersect the function once. It is a fact that if a function is one-to-one on its domain, then it has an inverse there.

Example: Find the inverse of $y = \frac{x-1}{2x+7}$.

solution: The way to find the inverse of a function is to have x and y "switch places" and then solve for y. Thus we solve:

$$x = \frac{y-1}{2y+7} \Rightarrow x(2y+7) = y - 1$$
$$2xy + 7x = y - 1$$
$$2xy - y = -7x - 1$$
$$y(2x-1) = -7x - 1$$
$$y = \frac{-7x-1}{2x-1} = \frac{7x+1}{1-2x}$$

Thus if $f(x) = \frac{x-1}{2x+7}$, then $f^{-1} = \frac{7x+1}{1-2x}$.

1.5 Limits

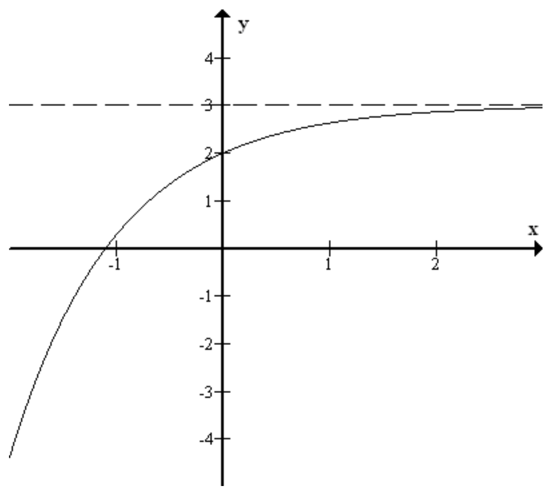
Definition 1.1 *If $f(x)$ becomes arbitrarily close to the number L as x approaches c from both sides, then we write*

$$\lim_{x \to c} f(x) = L$$

and say "the limit of $f(x)$ as x approaches c is L.

Note in the above that c could be ∞.

Intuition:



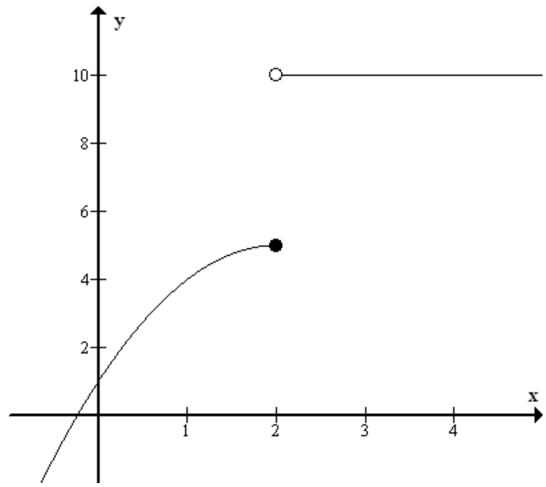
$$\lim_{x \to \infty} f(x) = 3$$

One-sided limits

- $\lim_{x \to c^-} f(x) = L$ – limit from the left.
- $\lim_{x \to c^+} f(x) = L$ – limit from the right.

In these cases you just approach from one side or the other.

Example: Suppose $f(x)$ has the following graph:



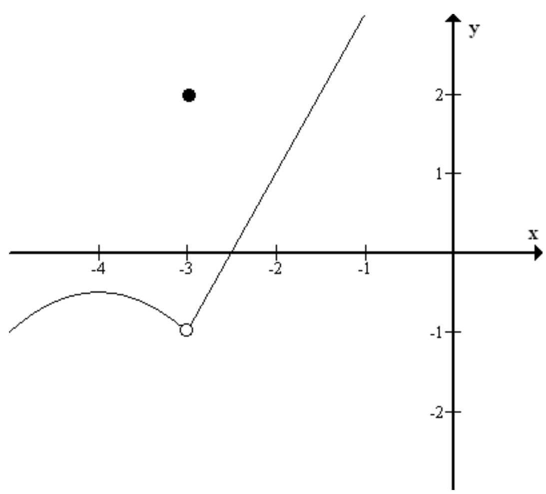
Note the closed and open circles. In this case, they mean that the value of f at 2 is 5. Here,

$$\lim_{x\to 2^+} f(x) = 10, \quad \lim_{x\to 2^-} f(x) = 5$$

If both the left and right limits exist and are equal, then $\lim_{x\to c} f(x) = L$

If they both exist but are not equal, then the ordinary (or two-sided) limit does not exist. So in the above picture, $\lim_{x\to 2} f(x)$ does not exist.

Example: Suppose $f(x)$ has the following graph:



Here, $\lim_{x\to -3^+} f(x) = -1$ and $\lim_{x\to -3^-} f(x) = -1$. Why? We don't care about the value of f at -3, only about values near -3. So, since $\lim_{x\to -3^+} f(x)$ and $\lim_{x\to -3^-} f(x)$ both exist and are equal, we have $\lim_{x\to -3} f(x) = -1$. (Even though $f(-3) = 2$!)

Frequently, it is the case that $\lim_{x\to c} f(x) = f(c)$. In other words, to calculate the limit, you can just plug c into the function. But, this is <u>not</u> always the case, as seen by the last example.

Examples:

1. Limit of a constant function:
$$\lim_{x\to 153} 5 = 5$$
Here the 5 on the left indicates the constant function $f(x) = 5$.

2. Limit of a linear function:
$$\lim_{x\to 2} 5x - 3 = 7$$
To see this, plug in numbers approaching 2:

x	$f(x)$
1.99	6.95
1.999	6.995
1.9999	6.9995
2.01	7.05
2.001	7.005
2.0001	7.0005

You can see that as x values approach 2 from either side, the value of f approaches 7. Note that in this case the limit is $f(2)$.

3. Evaluate the limit:
$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$
In this case, $f(3)$ is not defined. But,
$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = x+3 \quad \text{when } x \neq 3$$
Note that we are only allowed to cross out the $(x-3)$ term if $x \neq 3$. Thus for all values of x except 3, *even values very close to 3*, f behaves like the linear function $x+3$. Thus
$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

Operations with limits

Let b and c be real numbers and n a positive integer. Then

1. $\lim_{x\to c} b \cdot f(x) = b \lim_{x\to c} f(x)$
2. $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
3. $\lim_{x\to c} [f(x) \cdot g(x)] = \left[\lim_{x\to c} f(x)\right] \left[\lim_{x\to c} g(x)\right]$
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, assuming $\lim_{x\to c} g(x) \neq 0$
5. $\lim_{x\to c} x = c$
6. $\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$
7. $\lim_{x\to c} [f(x)]^{1/n} = \left[\lim_{x\to c} f(x)\right]^{1/n}$ assuming the nth root exists

From these it is easy to see that if $f(x)$ is any polynomial, then $\lim_{x\to c} f(x) = f(c)$.

Note how much easier it is to calculate limits with the formula $\lim_{x\to c} f(x) = f(c)$ than with a table.

Examples:

1. Evaluate $\lim_{x\to 2}(3x^3 - 5x^2 - 8x + 2)$.

   solution:
   $$\lim_{x\to 2}(3x^3 - 5x^2 - 8x + 2) = 3(2)^3 - 5(2)^2 - 8(2) + 2 = -10$$

2. Evaluate $\lim_{x\to -1} \frac{4x-5}{3-x}$

   solution:
   $$\lim_{x\to -1} \frac{4x-5}{3-x} = \frac{\lim_{x\to -1}(4x-5)}{\lim_{x\to -1}(3-x)} = \frac{4(-1)-5}{3-(-1)} = \frac{-9}{4}$$

3. Evaluate $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

   solution: In this case, I cannot just plug in 1 because that would make the denominator 0. But note that when we plug 1 into the numerator we also get 0. This means that $x-1$ must be a factor of x^3-1.

   In other words we can write $x^3 - 1 = (x-1)h(x)$ where $h(x)$ is some polynomial.

   To find $h(x)$ we do long division

$$
\begin{array}{r|l}
 & \quad x^2 + x + 1 \\
\hline
x - 1 & \quad x^3 - 1 \\
 & -(x^3 - x^2) \\
\hline
 & \qquad x^2 - 1 \\
 & \quad -(x^2 - x) \\
\hline
 & \qquad x - 1 \\
 & \quad -(x - 1) \\
\hline
 & \qquad\quad 0
\end{array}
$$

So, $x^3 - 1 = (x-1)(x^2+x+1)$. Thus,

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$$

4. Show that

$$\lim_{x\to 0} \frac{|3x|}{x}$$

does not exist.

<u>*solution:*</u> To do this, we show that the two one-sided limits are different.

First, let's compute $\lim_{x\to 0^-}$. Thus we are considering negative values of x. For negative values of x, $|3x| = -3x$, so

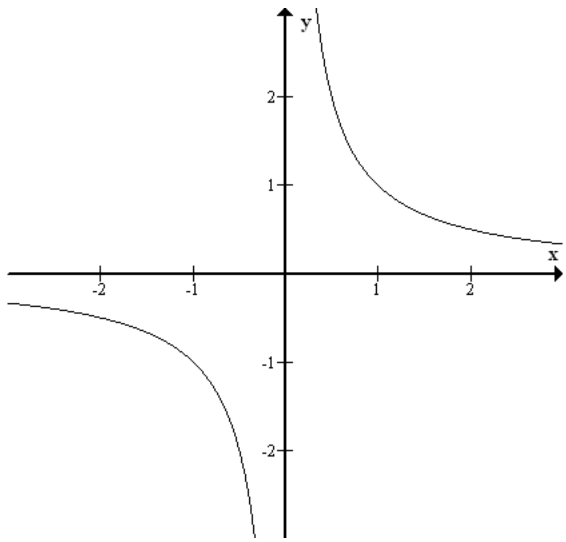
$$\lim_{x\to 0^-} \frac{|3x|}{x} = \lim_{x\to 0^-} \frac{-3x}{x} = \lim_{x\to 0^-} -3 = -3$$

Now, let's compute $\lim_{x\to 0^+}$. In this case we are considering positive x, so $|3x| = 3x$. Thus,

$$\lim_{x\to 0^+} \frac{|3x|}{x} = \lim_{x\to 0^+} \frac{3x}{x} = \lim_{x\to 0^-} 3 = 3$$

The two are different, so the limit doesn't exist.

5. Find the limits of $\frac{1}{x}$ as x approaches $\pm\infty$.



From the graph (and our intuition) we can see that the function approches 0 as x approaches $\pm\infty$. In other words,

$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$$

When the $\lim_{x\to\infty} f(x) = L$, we say that f has a **horizontal asymptote** at $y = L$. So $\frac{1}{x}$ has a horizontal asymptote at $y = 0$.

Note that

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Strictly speaking, these limits don't exist because $\pm\infty$ are not numbers. Instead, whenever we write that a limit as x approaches c is $\pm\infty$, we are saying that the function is "unbounded" as $x \to c$. When this happens, we say that $f(x)$ has a **vertical asymptote** at $x = c$.

1.6 Continuity

Recall that if $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$. These types of limits are easy to calculate. This leads to the following definition.

Definition 1.2 *Let $c \in (a, b)$ and $f(x)$ a function whose domain contains (a, b). then the function $f(x)$ is* ***continuous at*** *c if*

$$\lim_{x\to c} f(x) = f(c).$$

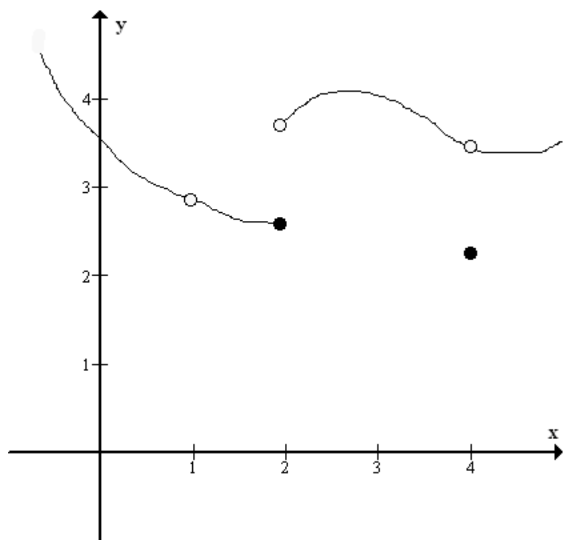
Note that this implies

1. $f(c)$ is defined,
2. the limit exists, and
3. the two are equal.

Intuition:

The graph of a continuous function is one that has no holes, jumps, or gaps. It can be "drawn without lifting the pencil". **This is intuition only**.

Example:



$f(x)$ is not continuous at

1. $x = 1$ because $f(1)$ is not defined.
2. $x = 2$ because $\lim_{x \to 2}$ does not exist.
3. $x = 4$. Here the limit exists, but is not equal to $f(4)$.

These are the three basic ways something can fail continuity.

Examples:

1. Any polynomial $p(x)$ is continuous everywhere.
2. A rational function is one of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. If $f(x)$ is a rational function, it will be continuous everywhere except where $q(x) = 0$ (in these places, $f(x)$ is undefined, hence certainly not continuous).

In general, if $f(x) = \frac{p(x)}{q(x)}$, where p and q are arbitrary, then $f(x)$ is continuous everywhere that p and q are continuous and q is not 0.

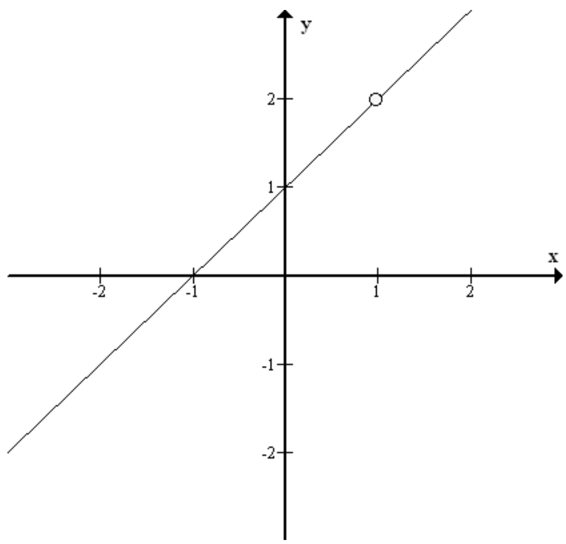
3. Consider

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Let's graph it. $f(x)$ is undefined at $x = 1$. If $x \neq 1$, we get

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1 \text{ – this is a line.}$$

So the graph is



This is continuous everywhere except $x = 1$. So the intervals on which it is continuous are $(-\infty, 1)$ and $(1, \infty)$.

4. The function

$$g(x) = \frac{1}{x^2 + 1}$$

is continuous everywhere because $x^2 + 1$ is never 0.

5. $h(x) = |x|$ is continuous everywhere. Why?

6. Consider

$$f(x) = \begin{cases} \frac{1}{2}x + 1 & x \leq 2 \\ 3 - x & x > 2 \end{cases}$$

This function is clearly continuous everywhere, except possibly at $x = 2$. Let's check $x = 2$.

$$\lim_{x \to 2} f(x) = ?$$

To calculate, we look at one-sided limits:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3 - x) = 1 \text{ (Why?)}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{1}{2}x + 1\right) = 2$$

So $\lim_{x \to 2} f(x)$ does not exist, so $f(x)$ is not continuous at $x = 2$.

7. Now consider

$$f(x) = \begin{cases} |x - 2| + 3 & x < 0 \\ x + 5 & x \geq 0 \end{cases}$$

This function is clearly continuous everywhere except possibly at $x = 0$. Let's check $x = 0$.

$$\lim_{x \to 0} f(x) = ?$$

Again, we look at one-sided limits:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 5) = 5$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (|x - 2| + 3) = 5$$

They are equal, so

$$\lim_{x \to 0} f(x) = 2 = f(0)$$

So f is continuous at $x = 0$, hence continuous everywhere.

4 Exponential and Logarithmic Functions

4.2 Exponential Functions and the Natural Base e

Note: we are not skipping 4.1 – this section of notes presents them both at the same time.

Definition 4.1 *If $a > 0$ and $a \neq 1$, then the exponential function with base a is given by $f(x) = a^x$.*

An important special case is when $a = e \approx 2.71828\ldots$, an irrational number.

<u>Properties of Exponents</u>

Let $a, b > 0$. Then,

1. $a^0 = 1$
2. $a^x a^y = a^{x+y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{a^y} = a^{x-y}$
6. $(\frac{a}{b})^x = \frac{a^x}{b^x}$
7. $a^{-x} = 1/a^x$

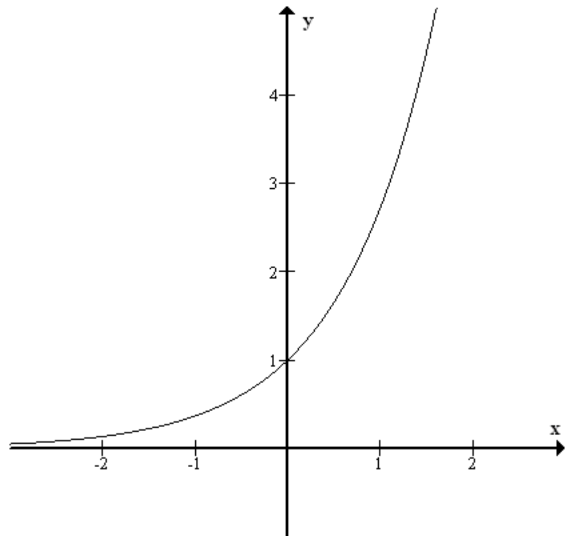
Definition 4.2 *The number e is defined to be*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim_{x\to 0}(1+x)^{1/x}$$

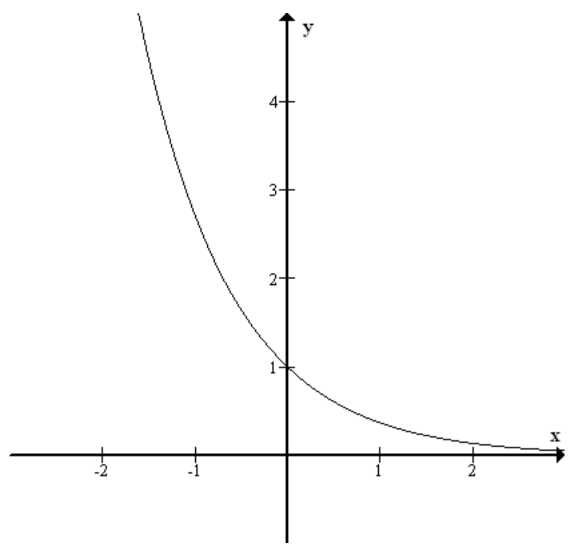
- It's possible to prove that this limit exists, but it's not obvious.
- It's an irrational number.

Graphs:

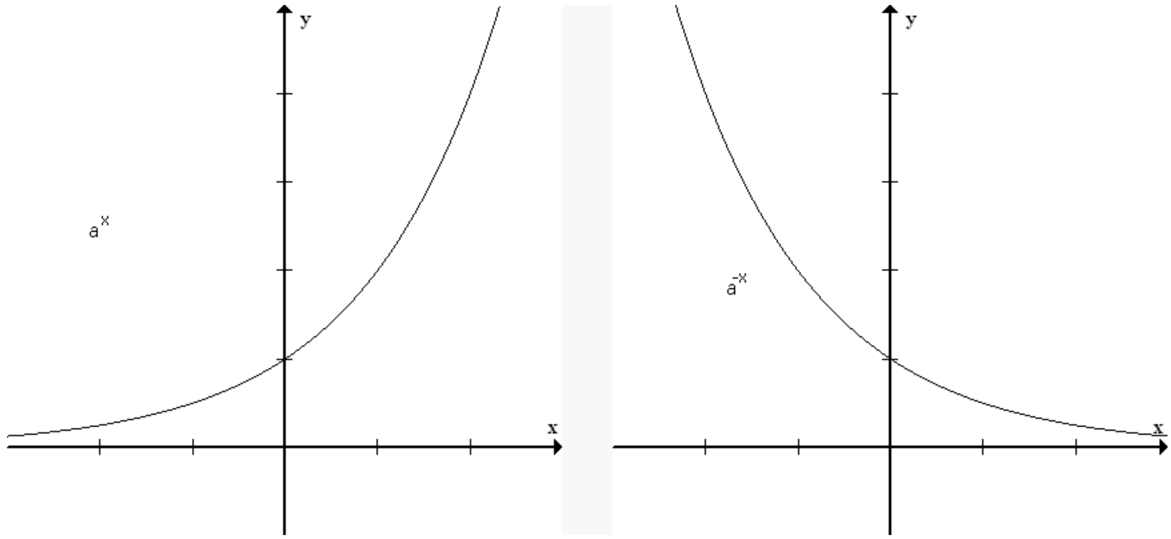
1. Exponential growth; e^x



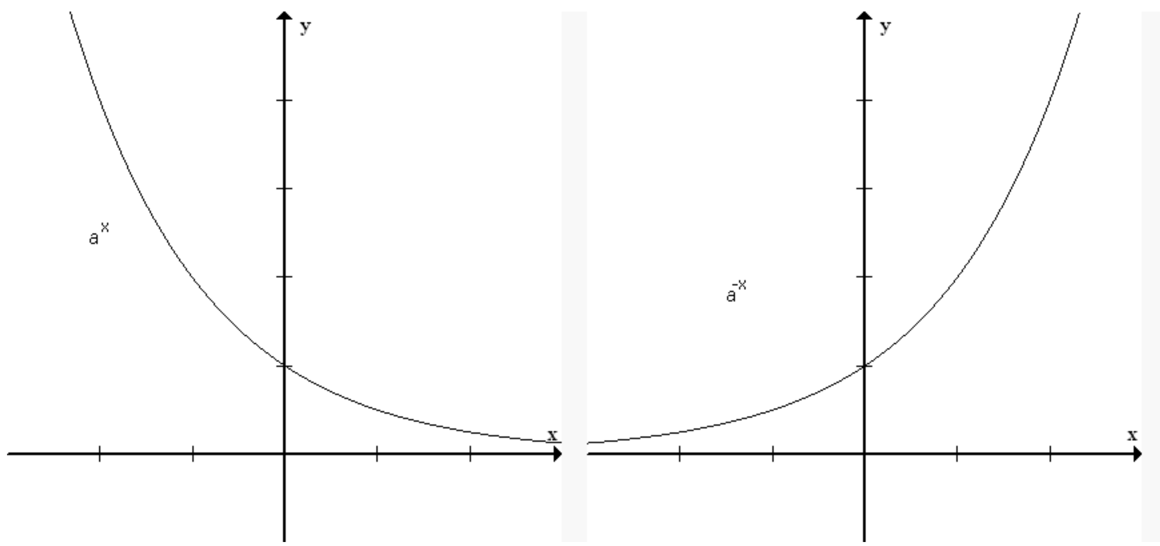
2. Exponential decay; e^{-x}



3. More generally, if $a > 0$,



If $a < 1$,



Here are some sample calculations you should be able to do with exponents (see p#297):

2. a)

$$\left(\frac{1}{3}\right)^3 = \frac{1}{5^3} = \frac{1}{125}$$

b)

$$\left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$$

c)

$$(64)^{2/3} = \left[(64)^{1/3}\right]^2 = 4^2 = 16$$

f)

$$4^{5/2} = \left[4^{1/2}\right]^5 = 2^5 = 32$$

You should be able to do these calculations on a test without the aid of a calculator.

Compounding Interest

Suppose I invest \$1 in a bank that pays 100% interest. Clearly, at the end of one year, I will have \$2 (it is also clear that I should be investing much more than \$1).

But suppose instead that after 6 months I withdraw my money and immediately re-invest it. How much money will I have at the end of the year?

After 6 months, we have \$1.50. If we then reinvest this at 100% interest for the rest of the year, we get

$$\$1.50\left(1 + \frac{1}{2}\right) = \$2.25$$

The $\frac{1}{2}$ term above corresponds to the interest rate (100% or 1.00) divided by the number of times we compounded in the year. So by getting interest on the interest we got from the first 6 months, we ended up with more money at the end of the year. What happens if we compound it more often?

Consider this table:

# of times compounded	\$ after 1 year (approx)
1	2
2	2.25
3	2.37
4	2.44
20	2.65
100	2.70
10000	2.72

Note that:

- The more times you compound, the more money you make. However, the amount of increase gets less and less.
- The numbers on the right hand side approach a limit. Can you see what it is?

The general formula: see p310. Let

$P =$ initial deposit (or *principal*),

$r =$ interest rate, expressed as a decimal,

$n =$ number of coumpoundings per year,

$t =$ number of years

Then the formula for $A(t)$, the balance after t years is

$$\boxed{A(t) = P\left[1 + \frac{r}{n}\right]^{nt}}$$

Why? You start with P dollars. When it is time for the first compounding, you multiply by $\left(1 + \frac{r}{n}\right)$(you would normally multiply by 1 plus the interest rate, but since only an nth of the year has passed we have to divide the interest rate by n). After each compounding, you multiply by another $\left(1 + \frac{r}{n}\right)$.

Example: Suppose you start with an initial deposit of \$2500 with an annual interest rate of 5%. How much do you have after 20 years if you compound yearly? Every 6 months? Every 3 months? Every month? Every day?

solution: The ns for each other above cases are $n = 1, 2, 4, 12, 365$. So the solutions can be filled into this chart:

n	$A(t)$
1	$2500(1 + 0.05)^{20} \approx 6633.24$
2	$2500\left(1 + \frac{.05}{2}\right)^{40}$
4	$2500\left(1 + \frac{.05}{4}\right)^{80}$
12	$2500\left(1 + \frac{.05}{12}\right)^{240}$
365	$2500\left(1 + \frac{.05}{365}\right)^{7300}$

Note: on tests you will not be expected to simplify these numbers.

Continuous Compounding

This process of compounding repeatedly has a limit at $n \to \infty$. This is called **continuous compounding**. Let's find this formula: we will do it for $t = 1$.

$$\begin{aligned} A &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n \\ &= P \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n \end{aligned}$$

If we let $x = r/n$, then $n = r/x$. Then as $n \to \infty$, $x \to 0$, so we have

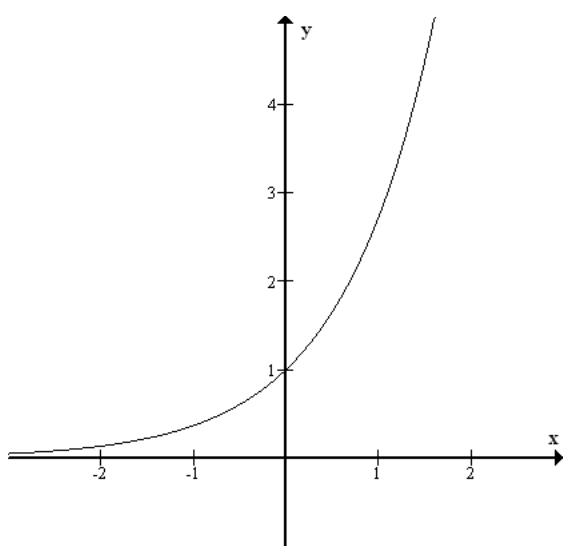
$$\begin{aligned} A &= P\lim_{x\to 0}(1+x)^{r/x} \\ &= P\left[\lim_{x\to 0}(1+x)^{1/x}\right]^r \\ &= P\cdot e^r \end{aligned}$$

In general, we have that if P is the inital amount and r is the rate, then with continuous compounding

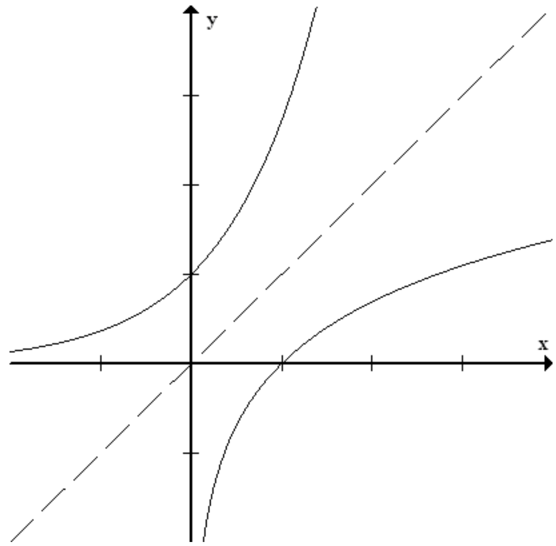
$$\boxed{A(t) = Pe^{rt}}$$

4.4 Logarithms

Note the graph of e^x



This graph passes the horizontal line test, so $f(x) = e^x$ is one-to-one and therefore has an inverse function. This is also true of $f(x) = a^x$ for any $a > 0, a \neq 1$.



More generally, for any $a > 1$ the graph of a^x and its inverse look like this. If $f(x) = a^x$, then we define the inverse function f^{-1} to be the **logarithm with base** a, and write

$$f^{-1}(x) = \log_a(x)$$

Note that, since the image of a^x is only the positive numbers, the domain of $\log_a(x)$ is all positive real numbers. The key property is:

$$\boxed{\log_a x = b \Longleftrightarrow a^b = x}$$

Examples:

$$\begin{array}{cc} \log_{10} 10 = 1 & 10^? = 10 \\ \log_5 25 = 2 & 5^? = 25 \\ \log_4 \frac{1}{2} = -\frac{1}{2} & 4^? = \frac{1}{2} \\ \log_5 \frac{1}{125} = -3 & 5^? = \frac{1}{125} \\ \uparrow & \uparrow \\ \text{log equation} & \text{corresponding} \\ & \text{exponential equation} \end{array}$$

Log Rules

1. Most important: by the properties of inverse functions we have

$$\log_b(b^x) = x \text{ and } b^{\log_b x} = x$$

The most important case of logs is when $b = e$. Log base e has a special name, in fact we define $\log_e x = \ln(x)$. So the above becomes

$$\boxed{\ln(e^x) = x \text{ and } e^{\ln(x)} = x}$$

LEARN THIS!!

The function $\ln(x)$ is known as the **natural logarithm function**, and $\ln(x)$ should be read as "the natural logarithm of x". In class, you may also hear me read this as "lawn x", but this isn't as standard.

Other rules: (I will state for ln, but they work for every log). Suppose that $x, y > 0$

2. $\ln(xy) = \ln(x) + \ln(y)$
3. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
4. $\ln(x^y) = y\ln(x)$

Calculations:

$$e^{3\ln(x)} = e^{\ln(x^3)} = x^3$$

$$\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1$$

Rewrite the following:

$$\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln(z) = \ln(x) + \ln(y) - \ln(z)$$

Compound Interest Revisited: (p324 #77)

A deposit of \$1000 is made in an account that earns interest at a rate of 5% per year. How long will it take for the balance to double if interest is compounded annually?

solution: From our earlier formula, our balance after t years is

$$A(t) = 1000\left(1 + \frac{.05}{1}\right)^t = 1000(1.05)^t$$

We are trying to find t such that $A(t) = 2000$. So we set the formula equal to 2000:

$$2000 = 1000(1.05)^t$$

$$2 = 1.05^t$$

We have to solve this for t. The general principle we use is that if we are trying to solve for a variable in the exponent, take log of both sides. So we get

$$\ln(2) = \ln(1.05^t)$$

$$\ln(2) = t\ln(1.05)$$

In this last point we see the point of using logs - the exponent can be brought down and solved for. So,

$$t = \frac{\ln(2)}{\ln(1.05)} \approx 14.21 \text{ years.}$$

Note that this is a very typical test/exam question. The answer I would expect is $t = \frac{\ln(2)}{\ln(1.05)}$.

Example 2: Same question, but now interest is compounded 10 times a year.

solution: By our formula,

$$A(t) = 1000\left(1 + \frac{.05}{10}\right)^{10t} = 1000(1.005)^{10t}$$

Again, we solve $A(t) = 2000$.

$$2000 = 1000(1.005)^{10t}$$
$$2 = 1.005^{10t}$$
$$\ln(2) = \ln(1.005^{10t})$$
$$\ln(2) = 10t\ln(1.005)$$

Therefore,

$$t = \frac{\ln(2)}{10\ln(1.005)} \approx 13.9 \text{ years.}$$

Example 3: Same question, but now interest is compounded continuously.

solution: By our formula,

$$A(t) = 1000e^{.05t}$$

Again, we solve $A(t) = 2000$.

$$2000 = 1000e^{.05t}$$
$$2 = e^{.05t}$$
$$\ln(2) = \ln(e^{.05t})$$
$$\ln(2) = .05t$$

Therefore,

$$t = \frac{\ln(2)}{.05} \approx 13.86 \text{ years.}$$

Notice that continuous compounding gives a kind of "best case scenario" – no amount of compounding will get your money to double faster than approximately 13.86 years.