University of Ottawa
MAT1330A Midterm Exam

November 16, 2016. Duration: 80 minutes. Instructor: Frithjof Lutscher.

Family Name:______ Solutions

First Name:______

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- Questions 3 and 6 are short answer questions. All other questions require complete legible solutions. You need to convince me that you know why your answer is correct.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: __________ out of 30

Question	1	2	3	4	5	6
Marks						

Question 1. [2 points] The function

$$f(x) = \begin{cases} \frac{x^2-1}{2x^2-x-1}, & x \neq 1, \\ k, & x = 1, \end{cases}$$

is continuous for $k = \boxed{}$.

$$\frac{x^2-1}{2x^2-x-1} = \frac{(x-1)(x+1)}{(x-1)(2x+1)}$$

Continuity: $\lim_{x \to 1} f(x) = f(1)$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x \to 1} \frac{x+1}{2x+1} = \frac{2}{3}$$

$f(1) = k$

$\Rightarrow$ Require $k = \frac{2}{3}$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 2t^3e^{-t/4}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

The infection peaks on day $\boxed{12}$

At the peak of the infection, there are $\boxed{2 \cdot 12^3 e^{-3} \text{ thousand}}$ people affected.

$$y'(t) = 2\left(3t^2e^{-t/4} + t^3e^{-t/4}\cdot\left(-\frac{1}{4}\right)\right) = 2t^2e^{-t/4}\left(3 - \frac{t}{4}\right)$$

$y'(t) = 0$ if $t = 0$ or $t = 12$

$$\left.\begin{array}{ll} y'(t) > 0 & \text{if } 0 < t < 12 \\ y'(t) < 0 & \text{if } t > 12 \end{array}\right\} \Rightarrow \text{global max at } t = 12$$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

The function f necessarily has a global minimum if...

$A)$ ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

$B)$ ...$f''(x) > 0$ for all x. true/false

$C)$ ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

$D)$ ...there exists a c such that $f'(c) = 0$ and f' changes sign at c true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^2)+e^{x^4}}{\sqrt{x}}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{1}{|x|}\left(\left(\frac{2}{x} + 4x^3 e^{x^4}\right)\sqrt{x} - \frac{1}{2\sqrt{x}}\left(\ln(x^2) + e^{x^4}\right)\right)$$

(b) $y(x) = x^{\cos(x)}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{d}{dx}\left(e^{\ln(x)\cos(x)}\right) = e^{\ln(x)\cos(x)}\left(\frac{\cos(x)}{x} - \ln(x)\sin(x)\right)$$

(c) $x^2y^2 = e^{2x+4y}$

larger!

$y'(x) = \boxed{}$

implicit: $2xy^2 + 2x^2yy' = e^{2x+4y}(2+4y')$

Solve for y': $(2x^2y - 4e^{2x+4y})y' = 2e^{2x+4y} - 2xy^2$

$$y' = \frac{2e^{2x+4y} - 2xy^2}{2x^2y - 4e^{2x+4y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+5x}-\sqrt{1+x^2}}{x} = \boxed{}$

$$= \lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1+x^2}}{x} \cdot \frac{\sqrt{1+5x}+\sqrt{1+x^2}}{\sqrt{1+5x}+\sqrt{1+x^2}}$$

$$= \lim_{x\to 0} \frac{1+5x-(1+x^2)}{x\left(\sqrt{1+5x}+\sqrt{1+x^2}\right)} = \lim_{x\to 0} \frac{5-x}{\sqrt{1+5x}+\sqrt{1+x^2}} = \frac{5}{2}$$

(b) $\lim\limits_{x\to\infty} \left(\ln(2x^2-2) - 2\ln(x+3)\right) = \boxed{}$

$$= \lim_{x\to\infty} \ln\left(\frac{2x^2-2}{(x+3)^2}\right) = \ln\left(\lim_{x\to\infty} \frac{2x^2-2}{x^2+6x+9}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{2-2/x^2}{1+6/x+9/x^2}\right) = \ln(2)$$

(c) $\lim\limits_{x\to0^+} \dfrac{\sqrt{3x}}{\sin(\sqrt{x})} = \boxed{}$

as $x \to 0^+$: $\sqrt{3x} \to 0$ and $\sin(\sqrt{x}) \to 0$

$\Rightarrow$ indeterminate form $\Rightarrow$ Apply L'Hopital

$$\lim_{x\to0^+} \frac{\sqrt{3x}}{\sin(\sqrt{x})} = \lim_{x\to0^+} \frac{\frac{3}{2\sqrt{3x}}}{\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}} = \lim_{x\to0^+} \frac{3\cdot 2\cdot\sqrt{x}}{2\cdot\sqrt{3x}\cdot\cos(\sqrt{x})}$$

$$= \lim_{x\to0^+} \frac{3}{\sqrt{3}\cdot\cos(\sqrt{x})} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(d) $\lim\limits_{x\to0} \dfrac{\cos(x)-1}{x^2} = \boxed{}$

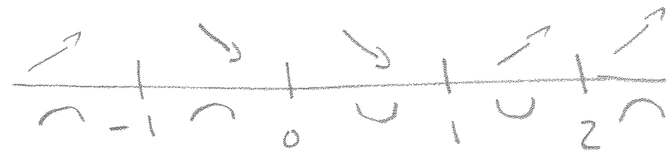
As $x \to 0$: $\cos(x)-1 \to 0$ and $x^2 \to 0$

$\Rightarrow$ indeterminate form $\Rightarrow$ Apply L'Hopital

$$\lim_{x\to0} \frac{\cos(x)-1}{x^2} \underset{\text{L'H}}{=} \lim_{x\to0} \underbrace{\frac{-\sin(x)}{2x}}_{\text{indeterminate again}} \underset{\text{L'H}}{=} \lim_{x\to0} \frac{-\cos(x)}{2} = -\frac{1}{2}$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| > 1$ and $f'(x) < 0$ if $|x| < 1$.
- $f''(x) < 0$ if $x < 0$ or $x > 2$ and $f''(x) > 0$ if $0 < x < 2$
- $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$
- $f(0) = 0$



Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

(i) The graph of $f(x)$ increasing for $|x| > 1$

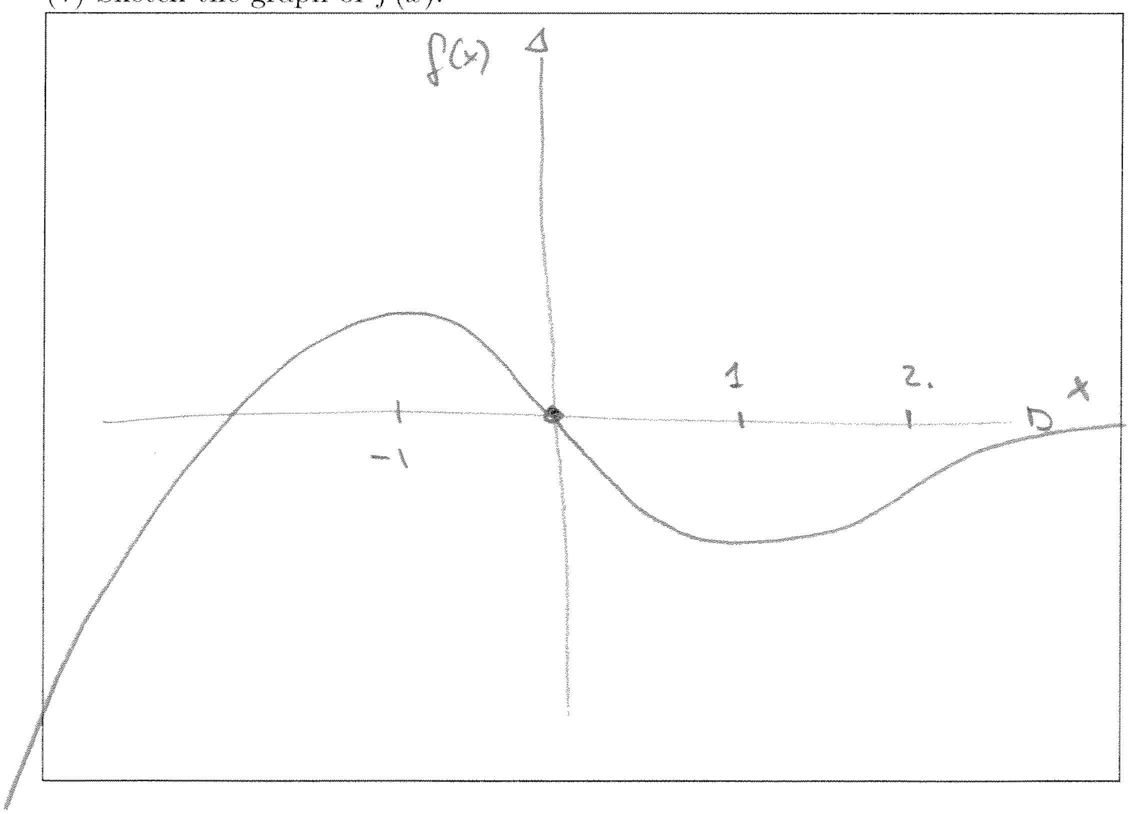
(ii) The graph of $f(x)$ concave up for $0 < x < 2$

re word

(iii) The graph of $f(x)$ attains a global maximum at -1

(iv) The graph of $f(x)$ attains a global minimum at none

(v) Sketch the graph of $f(x)$.



University of Ottawa
MAT1330A Midterm Exam

November 16, 2016. Duration: 80 minutes, Instructor: Frithjof Lutscher.

Family Name: Solutions

First Name: ____________________

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- Questions 3 and 6 are short answer questions. All other questions require complete legible solutions. You need to convince me that you know why your answer is correct.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: __________ out of 30

Question	1	2	3	4	5	6
Marks						

Question 1. [2 points] The function

$$f(x) = \begin{cases} \frac{2x^2-3x-2}{x^2-x-2}, & x \neq 2, \\ k, & x = 2, \end{cases}$$

is continuous for $k = \boxed{5/3}$.

$$\frac{2x^2-3x-2}{x^2-x-2} = \frac{(x-2)(2x+1)}{(x-2)(x+1)}$$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 3t^4e^{-t/4}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

The infection peaks on day $\boxed{16}$

At the peak of the infection, there are $\boxed{3 \cdot 16^4 \cdot e^{-4} \text{ thousand}}$ people affected.

$$y'(t) = 3t^3e^{-t/4}\left(4 - \frac{t}{4}\right)$$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

The function f necessarily has a global minimum if...

A) ...$f''(x) > 0$ for all x. true/false

B) ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

C) ...there exists a c such that $f'(c) = 0$ and f' changes sign. at c true/false

D) ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^3)+e^{x^2}}{\sqrt{x}}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{1}{|x|}\left(\left(\frac{3}{x} + 2xe^{x^2}\right)\sqrt{x} - \frac{1}{2\sqrt{x}}\left(\ln(x^3) + e^{x^2}\right)\right)$$

(b) $y(x) = x^{\tan(x)}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{d}{dx}\left(e^{\ln(x)\tan(x)}\right) = e^{\ln(x)\tan(x)}\left(\frac{\tan(x)}{x} + \ln(x)\sec^2(x)\right)$$

(c) $x^3y^2 = e^{3x+2y}$

$y'(x) = \boxed{}$

$$3x^2y^2 + 2x^3yy' = e^{3x+2y}(3+2y')$$

$$y' = \frac{3e^{3x+2y} - 3x^2y^2}{2x^3y - 2e^{3x+2y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim_{x \to 0} \frac{\sqrt{1+3x} - \sqrt{1+x^2}}{x} = \boxed{3/2}$

$$\frac{\left(\sqrt{1+3x} - \sqrt{1+x^2}\right)\left(\sqrt{1+3x} + \sqrt{1+x^2}\right)}{x\left(\sqrt{1+3x} + \sqrt{1+x^2}\right)}$$

$$= \frac{3 - x}{\sqrt{1+3x} + \sqrt{1+x^2}}$$

(b) $\lim_{x \to \infty} (\ln(4x^2 - 2) - 2\ln(x+1)) = \boxed{\ln(4)}$

$$\ln \frac{4x^2 - 2}{x^2 + 2x + 1}$$

(c) $\lim_{x\to 0^+} \dfrac{\sqrt{5x}}{\sin(\sqrt{x})} = \boxed{}$

$$\lim_{x\to 0^+} \frac{\sqrt{5x}}{\sin(\sqrt{x})} \underset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{5}{2\sqrt{5x}}}{\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{5 \cdot 2\sqrt{x}}{2\sqrt{5x}\,\cos(\sqrt{x})} = \lim_{x\to 0^+} \frac{5}{\sqrt{5}\cos(\sqrt{x})}$$

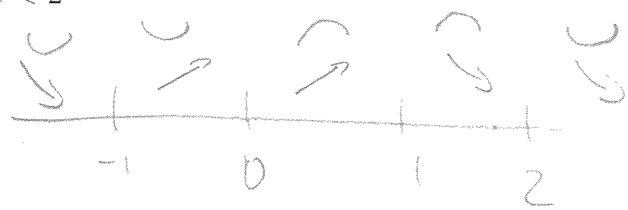
$$= \sqrt{5}$$

(d) $\lim_{x\to 0} \dfrac{x^2}{\cos(x)-1} = \boxed{}$

$$\lim_{x\to 0} \frac{x^2}{\cos(x)-1} \underset{\text{L'H}}{=} \lim_{x\to 0} \frac{2x}{-\sin(x)} \underset{\text{L'H}}{=} \lim_{x\to 0} \frac{2}{-\cos(x)} = -2$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| < 1$ and $f'(x) < 0$ if $|x| > 1$.
- $f''(x) > 0$ if $x < 0$ or $x > 2$ and $f''(x) < 0$ if $0 < x < 2$
- $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$
- $f(0) = 0$



Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

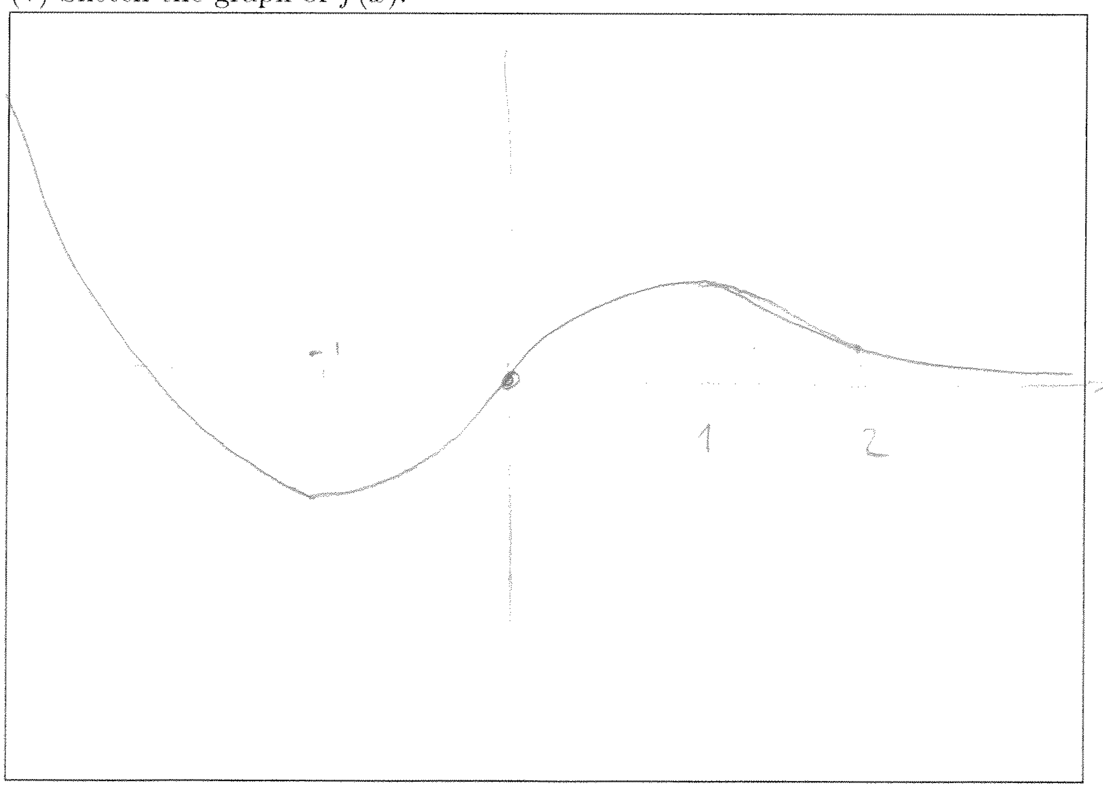
(i) The graph of $f(x)$ increasing for $\boxed{|x| < 1}$

(ii) The graph of $f(x)$ concave up for $\boxed{x < 0 \text{ or } x > 2}$

(iii) The graph of $f(x)$ attains a global maximum at $\boxed{\text{not at all}}$

(iv) The graph of $f(x)$ attains a global minimum at $\boxed{x = -1}$

(v) Sketch the graph of $f(x)$.



University of Ottawa
MAT1330A Midterm Exam

November 16, 2016. Duration: 80 minutes; Instructor: Frithjof Lutscher.

Family Name: Solutions

First Name: ______________________

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- Questions 3 and 6 are short answer questions. All other questions require complete legible solutions. You need to convince me that you know why your answer is correct.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: ___________ out of 30

Question	1	2	3	4	5	6
Marks						

Question 1. [2 points] The function

$$f(x) = \begin{cases} \frac{x^2-x-2}{2x^2-3x-2}, & x \neq 2, \\ k, & x = 2, \end{cases}$$

is continuous for $k = \boxed{3/5}$.

$$\frac{x^2-x-2}{2x^2-3x-2} = \frac{(x-2)(x+1)}{(x-2)(2x+1)}$$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 5t^4e^{-t/3}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

The infection peaks on day $\boxed{}$

At the peak of the infection, there are $\boxed{}$ people affected.

$5t^3e^{-t/3}\left(4 - \frac{t}{3}\right)$ $t = 12$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

The function f necessarily has a global minimum if...

$A)$ ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

$B)$ ...there exists a c such that $f'(c) = 0$ and f' changes sign at c true/false

$C)$ ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

$D)$ ...$f''(x) > 0$ for all x. true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^2)+e^{x^3}}{\sqrt{x}}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{1}{|x|}\left[\left(\frac{2}{x} + 3x^2 e^{x^3}\right)\sqrt{x} - \frac{1}{2\sqrt{x}}\left(\ln(x^2) + e^{x^3}\right)\right]$$

(b) $y(x) = x^{\sin(x)}$

$y'(x) = \boxed{}$

$$\frac{d}{dx} e^{\ln(x)\sin(x)} = e^{\ln(x)\sin(x)}\left(\frac{\sin(x)}{x} + \ln(x)\cos(x)\right)$$

(c) $x^2y^3 = e^{x+3y}$

$y'(x) = \boxed{}$

$$2xy^3 + 3x^2y^2y' = e^{x+3y}(1+3y')$$

$$y' = \frac{e^{x+3y} - 2xy^3}{3x^2y^2 - 3e^{x+3y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+4x}-\sqrt{1+x^2}}{x} = \boxed{2}$

$$\frac{\sqrt{1+4x}-\sqrt{1+x^2}}{x} \cdot \frac{\sqrt{1+4x}+\sqrt{1+x^2}}{\sqrt{1+4x}+\sqrt{1+x^2}}$$

$$= \frac{4x-x^2}{x\left(\sqrt{1+4x}+\sqrt{1+x^2}\right)} = \frac{4-x}{\sqrt{1+4x}+\sqrt{1+x^2}} \longrightarrow \frac{4}{2} = 2$$

(b) $\lim\limits_{x\to\infty} \left(\ln(5x^2+3) - 2\ln(x-1)\right) = \boxed{\ln(5)}$

$$\ln \frac{5x^2+3}{x^2-2x+1} = \ln \frac{5+3/x^2}{1-2/x+1/x^2}$$

(c) $\lim\limits_{x\to 0^+} \dfrac{\sqrt{6x}}{\sin(\sqrt{x})} = \boxed{}$

$$\lim_{x\to 0^+} \frac{\sqrt{6x}}{\sin(\sqrt{x})} \underset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{6}{2\sqrt{6x}}}{\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{6\cdot 2\sqrt{x}}{2\sqrt{6x}\,\cos(\sqrt{x})}$$

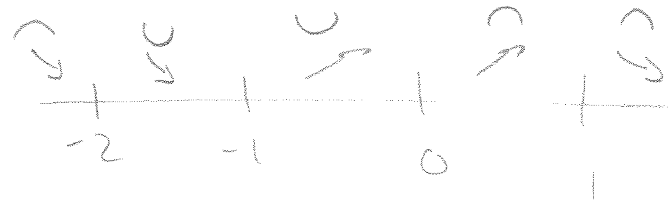
$$= \lim_{x\to 0^+} \frac{6}{\sqrt{6}\cos(\sqrt{x})} = \sqrt{6}$$

(d) $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos(x)} = \boxed{}$

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x)} \underset{L'H}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \underset{L'H}{=} \lim_{x\to 0} \frac{2}{\cos(x)} = 2$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| < 1$ and $f'(x) < 0$ if $|x| > 1$.
- $f''(x) < 0$ if $x < -2$ or $x > 0$ and $f''(x) > 0$ if $-2 < x < 0$
- $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = 0$
- $f(0) = 0$



Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

(i) The graph of $f(x)$ increasing for $\boxed{|x| < 1}$

(ii) The graph of $f(x)$ concave up for $\boxed{-2 < x < 0}$

(iii) The graph of $f(x)$ attains a global maximum at $\boxed{x = 1}$

(iv) The graph of $f(x)$ attains a global minimum at $\boxed{\text{not at all}}$

(v) Sketch the graph of $f(x)$.