S.B.

Carleton UNIVERSITY

FINAL EXAMINATION
December 2016

DURATION: 3 HOURS

Department Name & Course Number: **School of Mathematics & Statistics, MATH 1007A/B/C/D/E/F**

Course Instructor(s) Gary Bazdell, Luc Bourbonnais, Taqi Shatnawi, John Talboom, Steven Desjardins, Mathieu Lemire

AUTHORIZED MEMORANDA: NONE

NO CALCULATORS

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Question	Value	Grade
1	10	
2	7.5	
3	5	
4	3	
5	2	
6	4	
7	3	
8	14.5	
9	5.5	
10	3	
11	12	
12	3	
13	2.5	
Total	75	

1. [10 marks] Fill in the blanks. **Only the answer will be marked**.

a) $(-32)^{2/5} =$ 4

b) $\dfrac{d}{dx}\left(\displaystyle\int_0^{x^2} t\cos(e^t)\,dt\right) =$ $2x \; x^2 \cos(e^{x^2})$

c) $\log_4 64 =$ 3

d) $\lim\limits_{x\to\infty}\left(\dfrac{2x^3+x^2-2x+5}{9+7x-5x^3-2x^4}\right) =$ 0

e) $\sin^2(7)+\cos^2(7) =$ 1

f) $\int e^{-x}dx =$ $-e^{-x}+C$

g) Let $f(x)=\ln(5-3x)$. The domain of f = $(-\infty, 5/3)$ OR $\{x \in \mathbb{R} \mid x < 5/3\}$

h) $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{x}\right) =$ 2

i) $\dfrac{d}{dx}(\log_3 x) =$ $\dfrac{1}{(\ln 3)\cdot x}$

j) Let $f(x)=xe^x$. Then $f''(x) =$ $e^x(2+x)$ (Does not have to be in this form)

2. Let $f(x)=3x^4-4x^3-12x^2-4$.

a) [3] On what intervals is f i) increasing? ii) decreasing?

$f'(x) = 12x^3-12x^2-24x = 12x(x^2-x-2) = 12x(x-2)(x+1)$

$f'(x)=0 \Rightarrow 12x(x-2)(x+1)=0 \Rightarrow \begin{cases} x=-1 \\ x=0 \\ x=2 \end{cases}$ CRITICAL POINTS

From table below, we see that:

- f is increasing on $[-1, 0]$ and on $[2, \infty)$
- f is decreasing on $(-\infty, -1]$ and on $[0, 2]$

$-\infty$ $+\infty$

Could also just be "x"

		-1		0		2	
$12x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
f	↘	MIN	↗	MAX	↘	MIN.	↗

b) [1.5] Classify the critical points of f as local maxima, local minima, or neither.

From the table above, and using the first derivative test, we have:

$x=-1$ is a local min.
$x=0$ " " " max.
$x=2$ " " " min.

Question 2 continues on next page...

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c) [3] Find the absolute maximum and minimum of f on the interval $[-2,1]$.

End points $\begin{cases} f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 - 4 = 3(16) + 32 - 48 - 4 = 28 \\ f(1) = 3(1)^4 - 4(1)^3 - 12(1)^2 - 4 = 3 - 4 - 12 - 4 = -17 \end{cases}$

Relevant critical pts. $\begin{cases} f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 - 4 = 3 + 4 - 12 - 4 = -9 \\ f(0) = -4 \end{cases}$

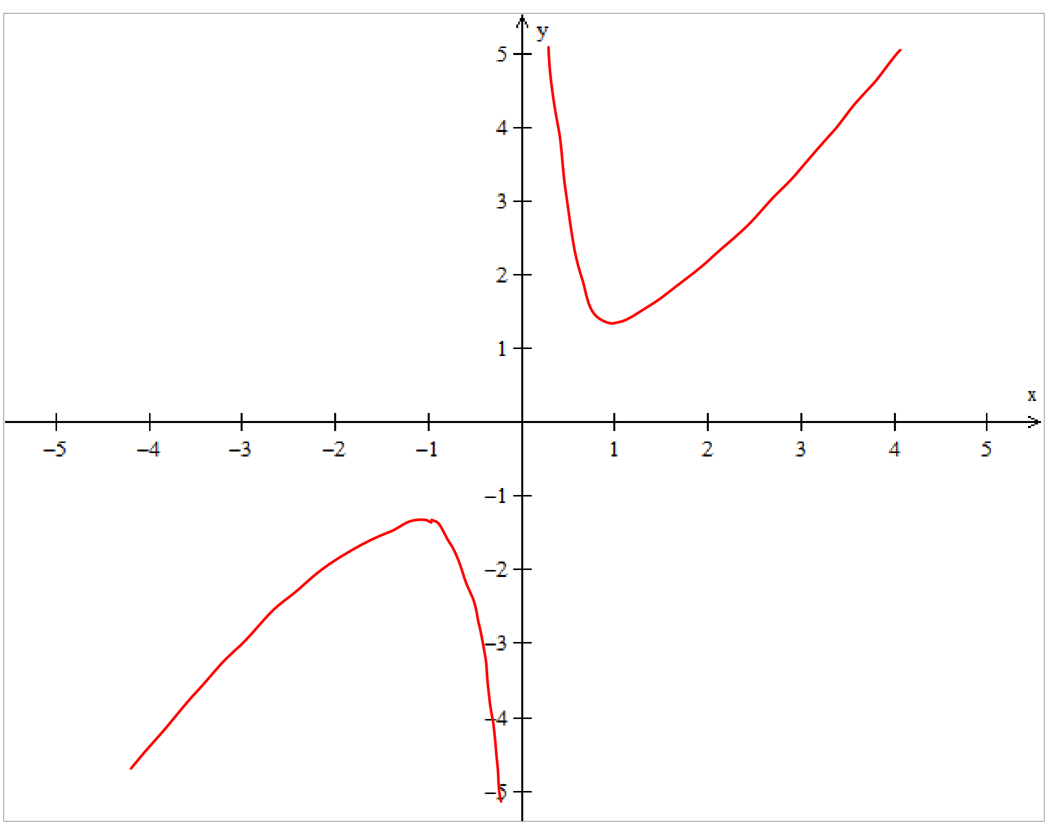
Abs. max. $= 28 = f(-2)$

Abs. min. $= -17 = f(1)$

3. [5] You are given the following information about the function $f(x) = x + \dfrac{1}{3x^3}$.

- Domain: $(-\infty, 0) \cup (0, \infty)$.
- Intercepts: none.
- Local maximum at $x = -1$ and local minimum at $x = 1$.
- f is increasing on $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 0)$ and $(0, 1)$.
- f is concave upward on $(0, \infty)$; concave downward on $(-\infty, 0)$.
- Vertical asymptote: $x = 0$; horizontal asymptotes: none.
- $\lim_{x \to -\infty} f(x) = -\infty$; $\lim_{x \to +\infty} f(x) = +\infty$.

Sketch the graph of f, showing all of the above information.



4. [3] Find the slope of the tangent line to the curve $x^4 + y^4 = 2xy$ at the point (1, 1).

$$\frac{d}{dx}\left(x^4 + y^4\right) = \frac{d}{dx}(2xy) = 2\frac{d}{dx}(xy)$$

$$4x^3 + 4y^3 y' = 2(y + xy') = 2y + 2xy'$$

$$4y^3 y' - 2xy' = 2y - 4x^3$$

$$y'(4y^3 - 2x) = 2y - 4x^3$$

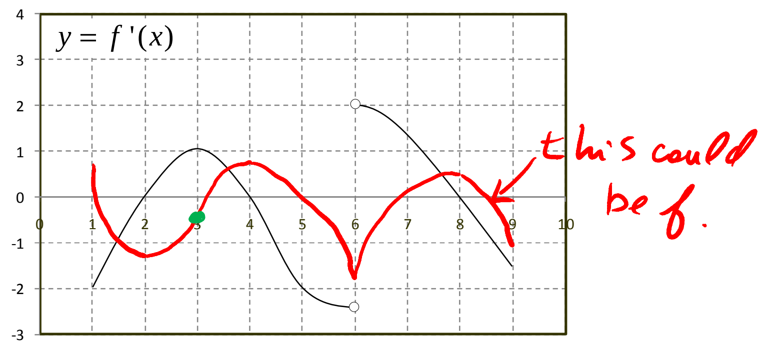
$$y' = \frac{2y - 4x^3}{4y^3 - 2x} \Rightarrow y'\Big|_{(1,1)} = \frac{2(1) - 4(1)^3}{4(1)^3 - 2(1)} = -1$$

5. [2] Let $h(x) = f(g(x))$. Find $h'(1)$ given: $g(1)=4$, $f'(4)=2$, $g'(1)=-5$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$\therefore \; h'(1) = f'(g(1)) \cdot g'(1) = f'(4) \cdot (-5) = (2)(-5) = -10$

6. The graph of the **derivative** f' of a continuous function f on the interval [1, 9] is shown.



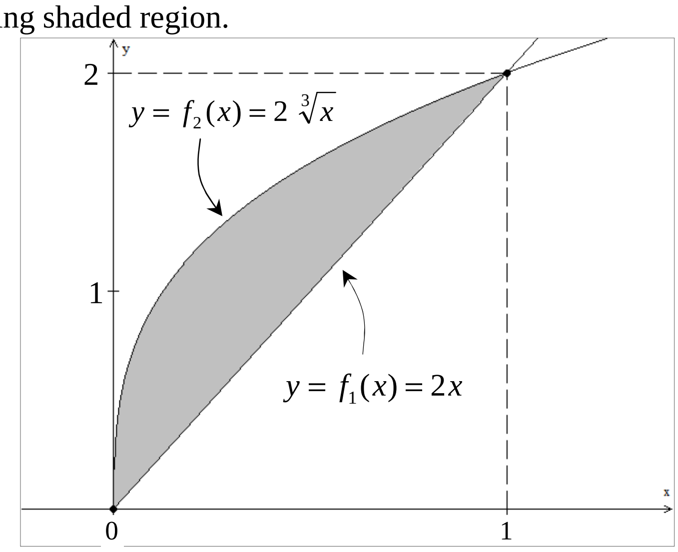
a) [2] List the critical points of f.

- $f'(x) = 0 \Rightarrow x = 2, 4, 8$
- $f'(x)$ undefined : $x = 6$

b) [2] On what intervals is f concave: i) upwards; ii) downwards?

- Concave upwards on $[1, 3]$, where f' is increasing
- Concave downwards on $[3, 9]$ where f' is decreasing

7. [3] Determine the area of the following shaded region.



$\langle AREA \rangle = \int_0^1 2\sqrt[3]{x} - 2x \, dx =$

$\ldots = \int_0^1 2x^{1/3} - 2x \, dx =$

$\ldots = \left[2 \frac{x^{4/3}}{4/3} - x^2 \right]_0^1 =$

$\ldots = \left[\frac{3}{2}(x^{4/3}) - x^2 \right]_0^1 =$

$\ldots = \frac{3}{2} - 1 = \frac{1}{2}$

8. Find the **derivative** of the following functions. **DO NOT SIMPLIFY**.

a) [3.5] $f(x)=\dfrac{x^2+\ln x}{\tan(2x)} = \dfrac{u}{v}$; $\begin{cases} u = x^2+\ln x \\ u' = 2x + 1/x \\ v = \tan(2x) \\ v' = 2\sec^2(2x) \end{cases}$

$\Downarrow$

$$f'(x) = \frac{u'v-uv'}{v^2}$$

$$= \frac{(2x+1/x)\tan(2x) - (x^2+\ln x)(2\sec^2(2x))}{\tan^2(2x)}$$

b) [3.5] $f(x)=e^{(x^2\sin x)} = e^{w}$; $w = uv$; $\begin{cases} u = x^2 \\ u' = 2x \\ v = \sin x \\ v' = \cos x \end{cases}$

$\Downarrow$

$$f'(x) = e^{w}\cdot w' = e^{w}(u'v+uv')$$

$$f'(x) = e^{x^2\sin x}\cdot(2x\sin x + x^2\cos x)$$

c) [3.5] $f(x)=(2x^3-5x^2)^4\ (x^5+7x)^7 = uv$; $\begin{cases} u = (2x^3-5x^2)^4 \\ u' = 4(2x^3-5x^2)^3(6x^2-10x) \\ v = (x^5+7x)^7 \\ v' = 7(x^5+7x)^6(5x^4+7) \end{cases}$

$\Downarrow$

$$f'(x) = u'v+uv'$$

$$f'(x) = 4(2x^3-5x^2)^3(6x^2-10x)(x^5+7x)^7 + (2x^3-5x^2)^4(7)(x^5+7x)^6(5x^4+7)$$

d) [4] $f(x)=(\cos x)^{(3x-5)}$ (Use logarithmic differentiation)

Set $y=(\cos x)^{(3x-5)}$

$$\ln y = \ln\left[(\cos x)^{(3x-5)}\right] = (3x-5)\cdot\ln(\cos x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left((3x-5)\cdot\ln(\cos x)\right)$$

$$\frac{1}{y}\cdot y' = 3\ln(\cos x) + (3x-5)\cdot\frac{1}{\cos x}\cdot(-\sin x)$$

$$y' = (\cos x)^{(3x-5)}\left[3\ln(\cos x) - (3x-5)\underbrace{\tan x}_{\frac{\sin x}{\cos x}}\right]$$

9. Determine the following integrals.

a) [2.5] $\int \frac{6}{x^2+1} - e^{-4x} + \frac{1}{2x+5} - 2\cos(7x)\, dx$

$$= 6\tan^{-1}x + \frac{1}{4}e^{-4x} + \frac{1}{2}\ln|2x+5| - \frac{2}{7}\sin(7x) + C_1$$

b) [3] $\int \frac{(x+3)^2}{x}\, dx = \int \frac{x^2+6x+9}{x} dx = \int x + 6 + \frac{9}{x}\, dx$

$$\ldots = \frac{x^2}{2} + 6x + 9\ln|x| + C_1$$

10. [3] Let $f'(x) = 12x^2 + 4x - 6$. Determine $f(x)$ if $f(1) = -2$.

$$f(x) = \int f'(x)\,dx = \int 12x^2 + 4x - 6\, dx$$

$$f(x) = 4x^3 + 2x^2 - 6x + C_1$$

but $f(1) = -2 = 4(1)^3 + 2(1)^2 - 6(1) + C_1$

$$-2 = 4 + 2 - 6 + C_1 \Rightarrow C_1 = -2$$

$$\therefore\ f(x) = 4x^3 + 2x^2 - 6x - 2$$

11. Evaluate the following limits. If a particular limit does not exist, then determine whether it is $+\infty, -\infty$, or neither. You may use L'Hospital's Rule when appropriate.

a) [2.5] $\lim_{x\to 3^-}\left(\frac{-3x^3}{x-3}\right)$ $\xrightarrow{D.S.}$ $\frac{-3(3)^3}{3-3} = \frac{-81}{0}$, so infinite limit

$+\infty$ OR $-\infty$?

$$\Rightarrow \lim_{x\to 3^-}\left(\frac{-3x^3}{x-3}\right) \xrightarrow{D.S.} \frac{-3(3)^3}{3^- - 3} = \frac{-81}{0^-} \to \frac{(-)}{(-)} = (+)$$

$$\therefore \lim_{x\to 3^-}\left(\frac{-3x^3}{x-3}\right) = +\infty$$

b) [3] $\lim_{x\to 0}\left(\frac{e^{4x}-1-4x}{x^2}\right)$ $\xrightarrow{D.S.}$ $\frac{e^{4(0)}-1-4(0)}{0^2} = \frac{0}{0}$

$$\xrightarrow{H.} \lim_{x\to 0}\left[\frac{4e^{4x}-4}{2x}\right] \xrightarrow{D.S.} \frac{4e^{4(0)}-4}{2(0)} = \frac{0}{0}$$

$$\xrightarrow{H.} \lim_{x\to 0}\left[\frac{16e^{4x}}{2}\right] \xrightarrow{D.S.} \frac{16e^{4(0)}}{2} = 8$$

c) [4] $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$ $\xrightarrow{D.S.}$ $\infty - \infty$

$$\ldots = \lim_{x\to\infty}\left[\left(\sqrt{x^2+x}-x\right)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right] =$$

$$\ldots = \lim_{x\to\infty}\left[\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}\right] = \lim_{x\to\infty}\left[\frac{x}{\sqrt{x^2+x}+x}\right] =$$

$$\ldots = \lim_{x\to\infty}\left[\frac{x}{x\sqrt{1+1/x}+x}\right] = \lim_{x\to\infty}\left[\frac{1}{\sqrt{1+1/x}+1}\right] \xrightarrow{D.S.}$$

$$\ldots = \frac{1}{\sqrt{1+\frac{1}{\infty}}+1} = \frac{1}{2}$$

Question 11 continues on next page...

d) [2.5] $\lim_{x\to 2}\left(\dfrac{x^2+5x-14}{x-2}\right) \xrightarrow{D.S.} \dfrac{2^2+5(2)-14}{2-2} = \dfrac{0}{0}$

$$\ldots = \lim_{x\to 2}\left[\frac{\cancel{(x-2)}(x+7)}{\cancel{(x-2)}}\right] = \lim_{x\to 2}(x+7) \xrightarrow{D.S.} 9$$

12. [3] Determine the linearization of $f(x)=\sqrt{9+x^2}$ at $x = 4$.

$L(x) = f(x_0) + f'(x_0)\cdot(x-x_0)$, with:

$x_0 = 4$

$f(x_0) = f(4) = \sqrt{9+4^2} = 5$

$f'(x) = \dfrac{x}{\sqrt{9+x^2}} \Rightarrow f'(x_0) = f'(4) = \dfrac{4}{\sqrt{9+4^2}} = \dfrac{4}{5}$

$\therefore\ y = L(x) = 5 + \frac{4}{5}(x-4)$

13. [2.5] Determine the inverse of $f(x)=\dfrac{(3x+8)^3}{2}-5$.

$y = \dfrac{(3x+8)^3}{2} - 5 \quad (= f(x))$

$x = \dfrac{(3y+8)^3}{2} - 5 \quad (= f(y))$

$2(x+5) = (3y+8)^3$

$3y+8 = \sqrt[3]{2x+10}$

$3y = \sqrt[3]{2x+10} - 8$

$y = \dfrac{\sqrt[3]{2x+10} - 8}{3} \quad (= f^{-1}(x))$

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