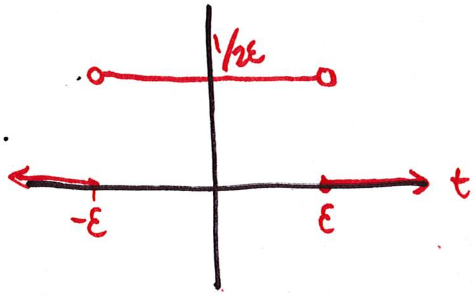
8.8 The Dirac Delta

Another function that is often seen in applications is an **i**mpulse function, representing an **i**nstantaneous behaviour at a particular moment of time. To create this function, we first define:

$$\delta_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & \text{if } -\epsilon < t < \epsilon \\ 0, & \text{if } |t| \geq \epsilon \end{cases}.$$



Note that the area under $\delta_\epsilon(t)$ is always equal to 1, which we can show pretty easily:

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\, dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\, dt = \frac{1}{2\varepsilon} t \Big|_{-\varepsilon}^{\varepsilon}$$

$$= \frac{1}{2\varepsilon}(\varepsilon - (-\varepsilon))$$

$$= \frac{2\varepsilon}{2\varepsilon} \boxed{= 1}$$

As ϵ decreases, the height of the step **i**ncreases, while the width **d**ecreases; but the area underneath is always 1. We may decrease ϵ arbitrarily, and even speak of a limit. Let

$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t),$$

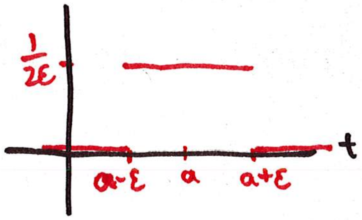
with the property that $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$. Then, we call this limit the **D**irac **D**elta or **u**nit **i**mpulse at $t = 0$.

✍ It is not quite correct to speak of the Dirac Delta as a function, but rather it is the result of a limit involving functions. For many applications, it will be safe for us to assume that it behaves generally like a function, however.

For example, the Dirac delta follows the horizontal translation property of functions: That is, $\delta(t - a)$ is therefore a unit impulse at time $t = a$.

Example 12. *Find the Laplace transform of the Dirac Delta $\delta(t-a)$, where $a > 0$.*

We define : $\mathcal{L}\{\delta(t-a)\} = \lim\limits_{\varepsilon\to 0} \mathcal{L}\{\delta_\varepsilon(t-a)\}$



$$= \lim_{\varepsilon\to 0} \int_0^\infty e^{-st}\,\delta_\varepsilon(t-a)\,dt$$

$$= \lim_{\varepsilon\to 0} \int_{a-\varepsilon}^{a+\varepsilon} e^{-st}\left(\frac{1}{2\varepsilon}\right) dt$$

$$= \lim_{\varepsilon\to 0} \frac{-1}{2\varepsilon s}\, e^{-st}\Big|_{a-\varepsilon}^{a+\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{-1}{2s\varepsilon}\left[e^{-s(a+\varepsilon)} - e^{-s(a-\varepsilon)}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{-1}{2s\varepsilon}\, e^{-as}\left[e^{-s\varepsilon} - e^{s\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{e^{-as}}{s\varepsilon}\Bigg[\underbrace{\frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2}}_{\sinh(s\varepsilon)}\Bigg] \quad !??!$$

$$= \lim_{\varepsilon\to 0} e^{-as}\,\frac{\sinh(s\varepsilon)}{s\varepsilon} \qquad \text{“}\frac{0}{0}\text{”}$$

L'Hôpital !

$$= \lim_{\varepsilon\to 0} e^{-as}\,\frac{\cancel{s}\overset{\nearrow 1}{\cosh(s\varepsilon)}}{\cancel{s}} = e^{-as}(1)$$

$$= e^{-as}$$

So we define

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

Dirac Deltas may also appear in DEs. If they do, use Laplace transforms to deal with them!

Example 13. *Solve the IVP given by*

$$\begin{aligned} y'' + 4y &= \delta(t-2\pi) \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Take Laplace Transforms Across:

$$s^2\mathcal{L}\{y\} - s\overset{=1}{y(0)} - \cancel{y'(0)} + 4\mathcal{L}\{y\} = e^{-2\pi s}$$

$$\mathcal{L}\{y\}\left[s^2+4\right] = e^{-2\pi s} + s$$

$$\mathcal{L}\{y\} = \frac{1}{2}e^{-2\pi s}\left(\frac{2}{s^2+4}\right) + \frac{s}{s^2+4}$$

$$y = \frac{1}{2}u(t-2\pi)\sin(2(t-2\pi)) + \cos(2t)$$

Note, $\sin(2(t-2\pi)) = \sin(2t)$ by trig identities.

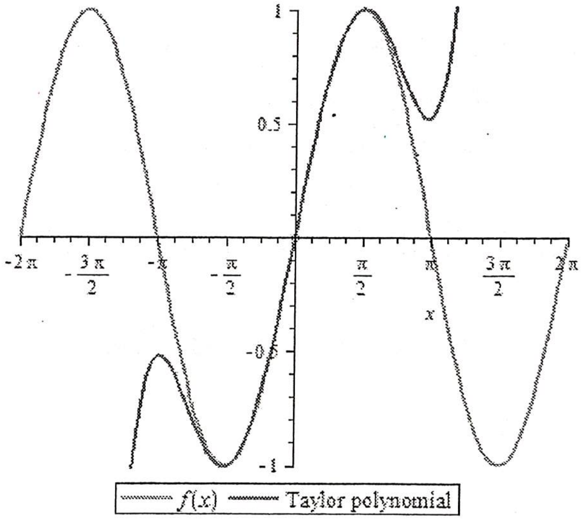
~~[illegible]~~

$$y = \cos(2t) + u(t-2\pi)\left[\frac{1}{2}\sin(2t)\right].$$

9 Using power series to solve DEs

Now, we will change gears and discuss an entirely different way to solve a DE. From first-year calculus, recall that functions like $\sin(x)$ or e^x or $\arctan(x)$, among many others, can be represented by an **i**nfinite **p**olynomial, called its **T**aylor **e**xpansion.

There were a few details that were left out in that first glance, like the **r**adius **o**f **c**onvergence, which tells us where the function is **e**qual to its expansion. Some of you have now learned more details about infinite series in Math*2200.



How can this concept be applied to differential equations, though?

While some solutions to DEs may be **e**xtremely **d**ifficult or **i**mpossible to find or to write down using the approaches we have looked at so far, we may be able to find an **i**nfinite **p**olynomial that satisfies the DE.

Suppose that we're working with a DE with dependent variable $y = y(x)$.

We start by assuming a form for the solution: We let y be an infinite polynomial in powers of x. For a polynomial centered at $x = p$, this means assuming that

$$y = \sum_{n=0}^{\infty} a_n (x-p)^n$$

The coefficients, a_n, determine the shape of y. Our job is to find these a_n coefficients that define *the* particular polynomial that satisfies the DE.

Let's see how this works by doing an example!

Consider the second-order DE:

$$y''(x) + y(x) = 0.$$

We already know from previously in this course that the general solution to this equation is

$y(x) =$ $C_1 \cos(x) + C_2 \sin(x)$. For now, though, we let y be a power series centered at $x = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

The coefficients a_n are to be determined. Assuming absolute convergence of this series in some interval around $x = 0$, we differentiate term-by-term twice and substitute into the DE. Taking these derivatives isn't hard!

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Sub in:

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{y''} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$$

To work forward a little more easily, it is convenient to take a step to s<u>hift</u> the i<u>ndex</u> on one of the sums so that the powers of x that appear in the two sums match one another. We obtain:

Let $n \rightarrow n-2$, in the second sum.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

Now as a single sum, factoring out the common power of x:

$$\sum_{n=2}^{\infty} \left[n(n-1)a_n + a_{n-2}\right] x^{n-2} = 0.$$

Notice that we still have a zero on the right-hand side. The only way that this polynomial could possibly be a boring old "zero" polynomial would be if *each* of the coefficients in front of the powers of x are equal to zero. So, it must be that:

$$(n)(n-1)a_n + a_{n-2} = \underline{\ 0\ }$$

for every integer $n \geq 2$.

We don't have any values of a_n figured out yet, so we must establish a **r**ecurrence **r**elation instead. This means that we start with **a**rbitrary **c**onstants a_0 and a_1, and find the other values of a_n in terms of a_0 and a_1. In this case,

$$a_n = \frac{-a_{n-2}}{n(n-1)}$$

So, given constants a_0 and a_1, we have:

$$\begin{aligned}
a_2 &= \frac{-a_0}{2(1)} = -\frac{1}{2}a_0 = -\frac{1}{2!}a_0 \\
a_3 &= \frac{-a_1}{3(2)} = -\frac{1}{6}a_1 = -\frac{1}{3!}a_1 \\
a_4 &= \frac{-a_2}{4(3)} = -\frac{1}{12}a_2 = -\frac{1}{12}\left(-\frac{1}{2}a_0\right) = \frac{1}{24}a_0 = \frac{1}{4!}a_0 \\
a_5 &= \frac{-a_3}{5(4)} = -\frac{1}{20}a_3 = -\frac{1}{20}\left(-\frac{1}{6}a_1\right) = \frac{1}{120}a_1 = \frac{1}{5!}a_1
\end{aligned}$$

... and so on!

What does this mean? Well, writing out our sum from the start, we have:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

Substituting in the coefficients we found:

$$y = a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 + \ldots$$

And grouping them by their respective arbitrary constants, we obtain:

$$y = a_0 \underbrace{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right]}_{\cos(x)} + a_1 \underbrace{\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots\right]}_{\sin(x)}.$$

These two series that form the solution may be familiar to you: They are the **T**aylor e xpansions for the cos and sin functions that we knew would be the solutions. The only difference is that we determined the solution using this very different approach.

✍ We can use power series to approach many more complicated DEs, however, that aren't easy to solve using other methods. This is a particularly useful method for linear DEs with variable coefficients, which we haven't dealt with much yet!

Example 1. *Find at least the first three terms of the power series solutions to:*

$$y'' + 2y' - xy = 0$$

Centre your power series at $x = 0$.

Assume:

$$y = \sum_{n=0}^{\infty} a_n x^n \longrightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Sub in:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n = 0.$$

Multiply in the x coeff:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Shift the sums to make sure the powers of x all match!

n-1 → n-2

n+1 → n-2

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=2}^{\infty} (n-1)a_{n-1} x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0.$$

These two sums have one extra "n=2" term that THIS sum doesn't have.

I write those two extra terms out front, then factor the common x powers out of the rest.

$$\underbrace{2(1)a_2x^0}_{\text{n=2 term from first sum.}} + 2(1)a_1x^0 + \sum_{n=3}^{\infty}\left[n(n-1)a_n + 2(n-1)a_{n-1} - a_{n-3}\right]x^{n-2} = 0.$$

(More space if necessary...)

We need the coefficient in front of every power of x to be equal to zero.

For the x^0 term, look outside the sum to those out-front terms.

Let a_0 and a_1 be arbitrary constants.

$$2a_2 + 2a_1 = 0 \longrightarrow \boxed{a_2 = -a_1}$$

For other powers, establish a recurrence relation, looking inside the sum.

$$n(n-1)a_n + 2(n-1)a_{n-1} - a_{n-3} = 0 \quad \text{for all } n \geq 3.$$

$$\rightarrow a_n = \frac{-2(n-1)a_{n-1} + a_{n-3}}{n(n-1)}, \quad n \geq 3.$$

(n=3)

$$a_3 = \frac{-2(2)a_2 + a_0}{3(2)} = -\frac{2}{3}a_2 + \frac{1}{6}a_0$$

$$= -\frac{2}{3}(-a_1) + \frac{1}{6}a_0$$

$$\boxed{= \frac{1}{6}a_0 + \frac{2}{3}a_1}$$

(n=4)

$$a_4 = \frac{-2(3)a_3 + a_1}{4(3)} = -\frac{1}{2}a_3 + \frac{1}{12}a_1$$

$$= -\frac{1}{2}\left(\frac{1}{6}a_0 + \frac{2}{3}a_1\right) + \frac{1}{12}a_1$$

$$\boxed{= -\frac{1}{12}a_0 - \frac{1}{4}a_1}$$

Remember, our solution was:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$= a_0 + a_1x + (-a_1)x^2 + \left(\frac{1}{6}a_0 + \frac{2}{3}a_1\right)x^3 + \left(-\frac{1}{12}a_0 - \frac{1}{4}a_1\right)x^4 + \dots$$

$$= a_0\left[1 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \dots\right] + a_1\left[x - x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + \dots\right]$$

Example 2. *Find at least the first three terms of the power series solutions to the IVP given by:*

$$\begin{aligned} 2y'' - (x-1)y &= 0 \\ y(1) &= 2 \\ y'(1) &= 1 \end{aligned}$$

Centre your power series at $x = 1$.

Assume:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \rightarrow y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$\rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Sub in:

$$2\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n = 0.$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} a_n(x-1)^{n+1} = 0.$$

Shift $n+1 \rightarrow n-2$

Shifting index:

$$2\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=3}^{\infty} a_{n-3}(x-1)^{n-2} = 0.$$

↑ I have one extra "n=2" term here. Write that one out front, then write the rest as a single sum.

$$2(2)(1)a_2(x-1)^0 + \sum_{n=3}^{\infty}\left[2n(n-1)a_n - a_{n-3}\right](x-1)^{n-2} = 0.$$

Every power of x-1 must have a zero coefficient to satisfy this equation!

(More space if necessary...)

Looking out front to the $(x-1)^0$ term, we have:

$$2(2)(1)a_2 = 0 \quad \rightarrow \quad \boxed{a_2 = 0}$$

Moving inside the sum, we need:

$$2(n)(n-1)a_n - a_{n-3} = 0 \qquad \text{for all } n \geq 3$$

(Let a_0 and a_1 be arbitrary constants.)

$$a_n = \frac{a_{n-3}}{2n(n-1)}$$

n=3 $$a_3 = \frac{a_0}{2(3)(2)} \boxed{= \frac{1}{12}a_0}$$

n=4 $$a_4 = \frac{a_1}{2(4)(3)} \boxed{= \frac{1}{24}a_1}$$

n=5 $$a_5 = \frac{a_2}{2(5)(4)} \boxed{= 0}$$ (since $a_2 = 0$ from above!)

n=6 $$a_6 = \frac{a_3}{2(6)(5)} = \frac{1}{60}\left(\frac{1}{12}a_0\right) \boxed{= \frac{1}{720}a_0}$$

n=7 $$a_7 = \frac{a_4}{2(7)(6)} = \frac{1}{84}\left(\frac{1}{24}a_1\right) \boxed{= \frac{1}{2016}a_1}$$

Our solution was:

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \ldots$$

$$y = a_0 + a_1(x-1) + \frac{1}{12}a_0(x-1)^3 + \frac{1a_1}{24}(x-1)^4 + \frac{1a_0}{720}(x-1)^6 + \frac{1}{2016}a_1(x-1)^7 + \ldots$$

$$y = a_0\left[1 + \frac{1}{12}(x-1)^3 + \frac{1}{720}(x-1)^6 + \ldots\right] + a_1\left[(x-1) + \frac{1}{24}(x-1)^4 + \frac{1}{2016}(x-1)^7 + \ldots\right]$$

We had ICs: $y(1) = 2, \quad y'(1) = 1.$

Sub them in: $y(1) = \boxed{2 = a_0}$

Also $y'(x) = a_0\left[\frac{1}{12}3(x-1)^2 + \frac{1}{720}6(x-1)^5 + \dots\right]$

$+ a_1\left[1 + \frac{1}{24}4(x-1)^3 + \frac{1}{2016}7(x-1)^6 + \dots\right].$

$\rightarrow \quad y'(1) = \boxed{1 = a_1}$

So the solution to the IVP is: (the same thing but with those a's subbed in.).

Consider the DE

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

where $A(x)$, $B(x)$, and $C(x)$ are polynomials with no common factors. Any value of x for which $A(x) \neq 0$ is called an o rdinary point. If $A(x^*) = 0$, however, then x^* is said to be a s ingular point. For example, the DE given by

$$(x-4)^2 y'' + (2 - x^2)y = 0$$

has a s ingular point at $x =$ 4 . For series solutions about singular points, we require techniques that can be quite a bit more involved.

To address this possibility and more, the discussion of power series solutions for DEs will continue in Math*3100 (Differential Equations II)!