## 8.4 Solving IVPs using Laplace transforms

We have all of the tools we need to proceed here. The idea is to transform a DE into an algebraic equation in terms of $s$ using all of the rules that we have presented so far, solve for the solution in terms of $s$, and then use inverse Laplace transforms to return to an answer in terms of the original variable.

**Example 17.** *Solve*

$$\begin{aligned} y''(t) + y(t) &= 0 \\ y(0) &= 2 \\ y'(0) &= 1 \end{aligned}$$

*by using Laplace transforms.*

Take the Laplace Transform across: $\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{0\}$

$$\underbrace{s^2\mathcal{L}\{y\} - sy(0) - y'(0)}_{\mathcal{L}\{y''\}} + \mathcal{L}\{y\} = 0$$

Factor out $\mathcal{L}\{y\}$, keeping them on the left:

$$\mathcal{L}\{y\}\underbrace{\left[s^2+1\right]}_{\text{The characteristic eq., but in } s!} = sy(0) + y'(0)$$

$$\rightarrow \mathcal{L}\{y\} = \frac{s}{s^2+1}y(0) + \frac{1}{s^2+1}y'(0)$$

Since we know these from the ICs, we can fill them in

$$\rightarrow \mathcal{L}\{y\} = 2\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$y = \mathcal{L}^{-1}\left\{2\frac{s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$\boxed{y = 2\cos(t) + \sin(t).}$$

**Example 18.** *Solve*

$$y'(t) + 4y(t) = \cos(2t)$$

$$y(0) = 1$$

*by using Laplace transforms.*

$$s\mathcal{L}\{y\} - \overset{=1}{y(0)} + 4\mathcal{L}\{y\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{y\}\big[\underbrace{s+4}_{\text{Char. Eq. but in s.}}\big] = \frac{s}{s^2+4} + 1$$

$$\rightarrow \quad \mathcal{L}\{y\} = \frac{s}{\underbrace{(s+4)(s^2+4)}} + \frac{1}{s+4}$$

Partial Fractions!

$$\frac{s}{(s+4)(s^2+4)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4}.$$

HSCU: $\boxed{A = -1/5}$

Then

$$\frac{s}{(s+4)(s^2+4)} = \frac{A(s^2+4) + (Bs+C)(s+4)}{(s+4)(s^2+4)}$$

Match Powers:

$s^2$: $\quad 0 = A + B \rightarrow \boxed{B = 1/5}$

Constants: $\quad 0 = 4A + 4C \rightarrow \boxed{C = 1/5}$

$$\rightarrow \mathcal{L}\{y\} = \frac{-1/5}{s+4} + \frac{1/5\, s + 1/5}{s^2+4} + \frac{1}{s+4}$$

$$\rightarrow \mathcal{L}\{y\} = \frac{4/5}{s+4} + \frac{1}{5}\frac{s}{s^2+4} + \frac{1}{10}\frac{2}{s^2+4}$$

Inverse Laplace:

$$\boxed{y = \frac{4}{5}e^{-4t} + \frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t)}$$

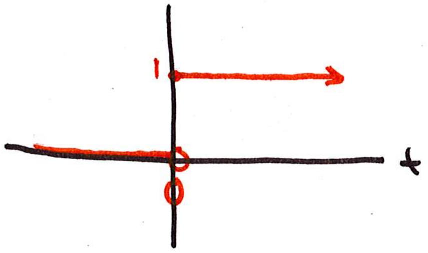
✍ We could check the answers to both this and the previous example by using techniques from earlier in the course. Try it!

## 8.5 Step Functions

Sometimes, we might encounter differential equations that have **d**iscontinuous right-hand sides. These sorts of DEs can be useful in modelling many types of systems, especially those featuring a **s**witch of some kind. We don't have any tools for yet, but we now introduce something that can help.

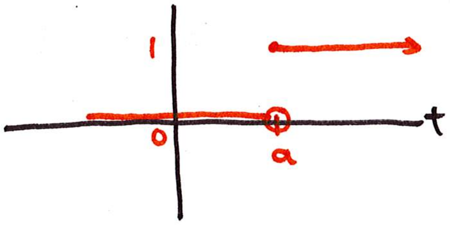
The **Heaviside step function** is defined by:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



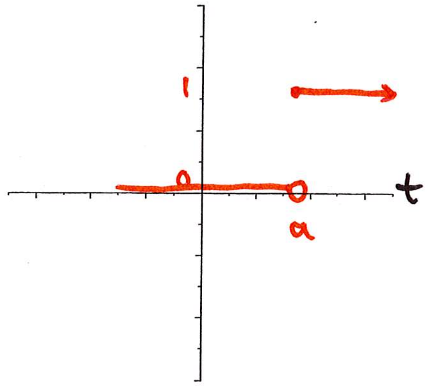
That means that for a constant $a$, we have that

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



✍ Depending on the source, $u(t-a)$ is sometimes denoted as $H(t-a)$ (H for **H**eaviside) or $u_a(t)$, as Boyce and DiPrima uses. Be flexible and ready to shift your thinking depending on the context if you see this function in other courses!

This is pretty easy to graph:



✍ Which branch includes the endpoint of the discontinuity usually doesn't matter in the context of DEs, where the important aspect is simply that a sudden change happens at that value. By convention, the step function is typically defined as we have.

We can find the Laplace transform of the Heaviside function just as we can for any other function. Let's do it!

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt$$

Split into two integrals!

$$= \cancel{\int_0^a e^{-st}(0)\,dt} + \int_a^\infty e^{-st}(1)\,dt$$

Area under 0 is zero!

$$= -\frac{1}{s}\left[e^{-st}\right|_a^\infty$$

$$= -\frac{1}{s}\lim_{b\to\infty}\left[\cancelto{0}{e^{-sb}} - e^{-sa}\right]$$

$$\boxed{= \frac{e^{-as}}{s}}$$

If $a = 0$, then $\mathcal{L}\{u(t-a)\} = \dfrac{1}{s}$. This should make sense. for $t \geq 0$, the Heaviside function is exactly the function $f(t) = 1$, so these two functions **s**hould have the **s**ame **L**aplace **t**ransforms !

## 8.6 Creating other Functions from the Heaviside Function

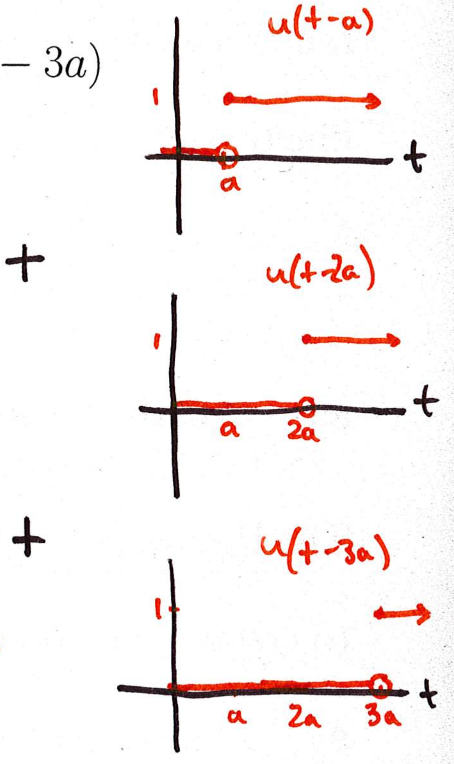
The Heaviside function is really interesting, because it can be used to create many other functions. To do this, we **a**dd or **s**ubtract two or more different Heaviside functions to or from one another.

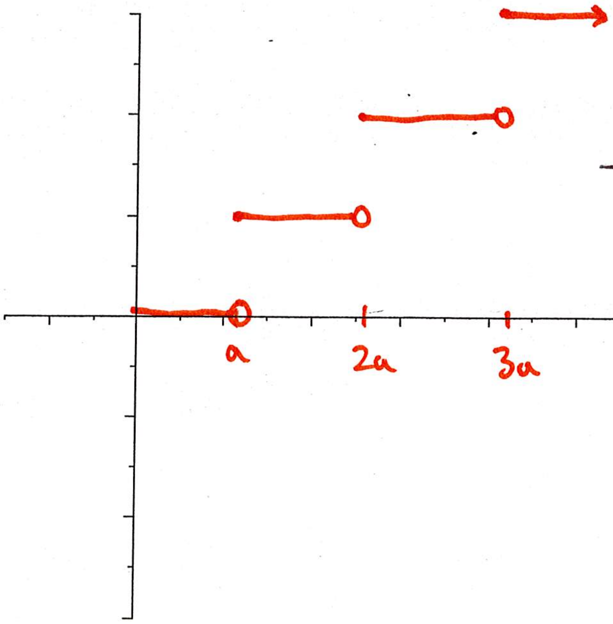
We'll explore a few examples on the next few pages.

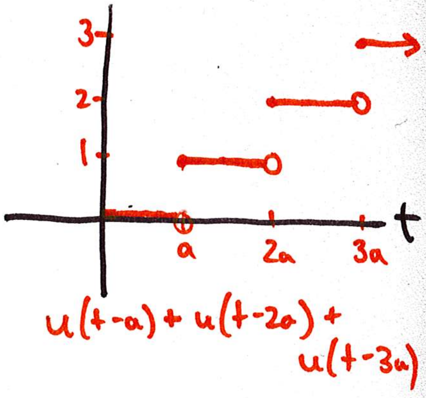
**Example 1.** *Let* $a > 0$ *and* $t \geq 0$. *Then, graph* $s_a(t)$, *given by:*

$$s_a(t) = u(t-a) + u(t-2a) + u(t-3a)$$

$$= \begin{cases} 0, & 0 \leq t < a \\ 1, & a \leq t < 2a \\ 2, & 2a \leq t < 3a \\ 3, & t \geq 3a \end{cases}$$



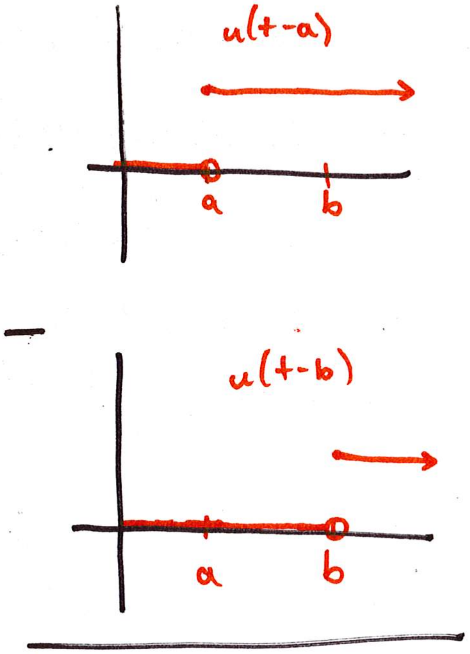


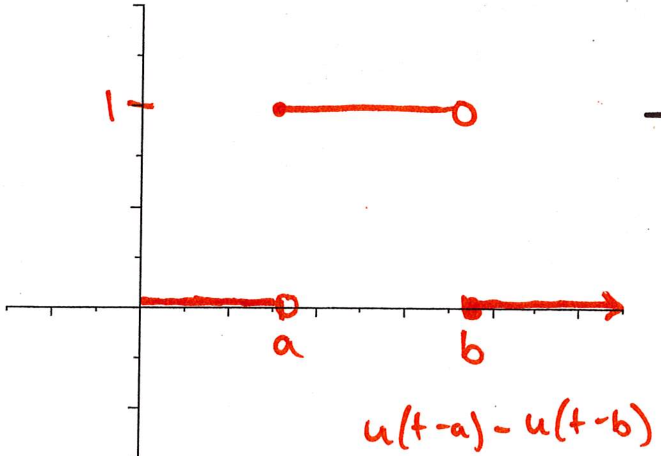


✍ This staircase function has three steps. We could easily create a staircase function with $n$ steps by adding together n Heaviside functions.

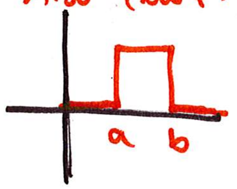
**Example 2.** *If $a < b$ and $t \geq 0$, then graph $r_{a,b}(t)$, given by:*

$$r_{a,b}(t) = u(t-a) - u(t-b) = \begin{cases} 0, & 0 \leq t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$$





Also (barf:)



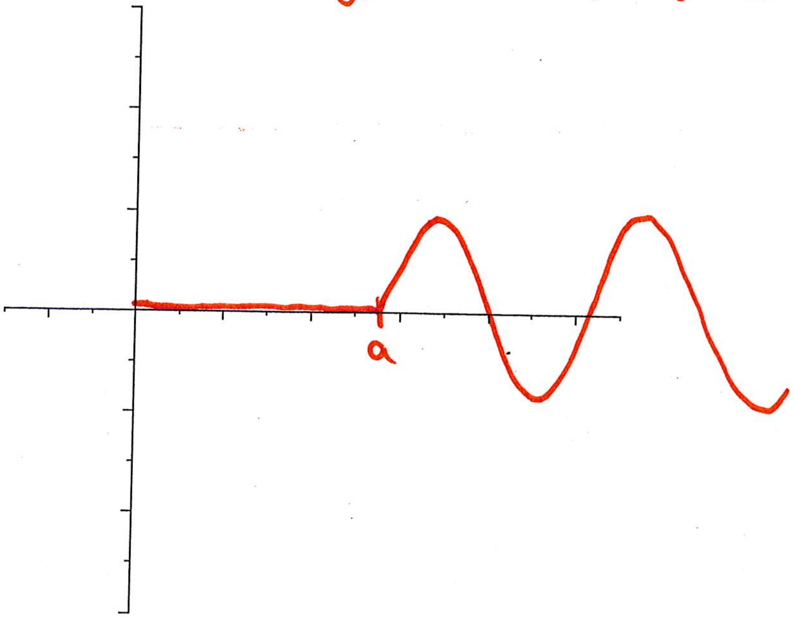
✍ This is a **r**ectangular **i**mpulse function.

## Turning a Function on:

**Example 3.** *Let $a > 0$ be the point of time at which we want a behaviour given by a function $f(t)$ to "switch on". We can define a function that does this using:*

$$\begin{aligned} g(t) &= u(t-a)f(t-a) \\ &= \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases} \end{aligned}$$

$g(t) = u(t-a)\sin(t-a)$



This comes up often enough that we'll even talk about its Laplace transform. We have that

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),$$

where $F(s) = \mathcal{L}\{f(t)\}$. (We could derive this from first principles!)

Going backwards, we therefore have

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \underline{\quad u(t-a)\, f(t-a) \quad},$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Taking the Laplace transforms of Heaviside functions can be a little bit tricky to get right, so we'll practice a bit before we continue.

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),$$

where

$$\mathcal{L}\{f(t)\} = F(s).$$

**Example 4.** *Evaluate* $\mathcal{L}\{u(t-2)\}$.

$$\frac{e^{-2s}}{s}$$

**Example 5.** *Evaluate* $\mathcal{L}\{(t-2)\underline{u(t-2)}\}$.

$$= e^{-2s}\left(\frac{1}{s^2}\right) \quad \text{or} \quad \frac{e^{-2s}}{s^2}.$$

**Example 6.** *Evaluate* $\mathcal{L}\{\underset{\sim}{t}u(t-2)\}$.

I need a $(t-2)$ to be multiplying our $u(t-2)$, but I only have $t$! So, MAKE them match, then adjust!

~~$= \mathcal{L}\{(t-2)u(t-2)\} + 2$~~

$$= \mathcal{L}\{(t-2)u(t-2) + 2u(t-2)\}$$

$$= \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}$$

**Example 7.** *Evaluate* $\mathcal{L}^{-1}\left\{\dfrac{e^{-6s}}{s^2+9}\right\}$. $= \mathcal{L}^{-1}\left\{e^{-6s}\dfrac{1}{s^2+9}\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-6s}\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin(3(t-6))\,u(t-6)$$

Functions with discontinuities may appear either in terms of Heaviside functions, or simply as a piecewise function. It would be handy if we could convert back and forth between these two formats. Luckily, this is not too complicated of a task.

## To write a piecewise function in terms of Heaviside functions:

Given a piecewise function:

- Write down the function appearing in the first branch.
- For each value "a" that separates two branches, one on the left and one on the right, add

  $u(t-a)$ [The right branch's function – The left branch's function] .

**Example 8.** *Write the following function as a combination of Heaviside step functions:*

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 2 \\ -t, & \text{if } 2 \leq t < 3 \cdot \\ 2t - e^t, & \text{if } t \geq 3 \end{cases}$$

$$1 + u(t-2)\Big[\underset{\substack{\uparrow \\ \text{Add} \\ \text{new} \\ \text{behaviour}}}{-t} \; \underset{\substack{\uparrow \\ \text{Subtract} \\ \text{off} \\ \text{old} \\ \text{behaviour}}}{-1}\Big] + u(t-3)\Big[\underbrace{2t - e^t}_{\substack{\uparrow \\ \text{Add} \\ \text{new}}} \underset{\substack{\uparrow \\ \text{Subtract} \\ \text{old.}}}{-(-t)}\Big],$$

$$(t \geq 0.)$$

## To write a combination of Heaviside functions as a piecewise function:

- Group the terms by the values $a_i$ in the Heaviside functions $u(t - a_i)$ that appear, in order of increasing $a_i$. Put any terms not multiplied by a Heaviside function before all of the others.
- Locate the first terms, those that are NOT multiplied by a Heaviside function. This is your first branch:

  Those terms,

  if $0 \leq t <$ (the lowest value of $a_i$ appearing in any term).

- Now, add a branch for each new term. Each branch should be of the form:

  The function for this term

  PLUS all previous functions, if $a_{previous} \leq t < a_{next-biggest}$.

- Continue until the last branch, which should be the sum of ALL of the functions appearing in each term, for $t \geq a_{biggest}$.

**Example 9.** *Write the following function as a single piecewise function:*

$$y(t) = \underline{40\sin(t)} + 25\cos(t-2)u(t-2) - 4tu(t-4) + 5\sin(t-4)u(t-4)$$

$$= 40\sin(t) + \left[25\cos(t-2)\right]u(t-2) + u(t-4)\left[-4t + 5\sin(t-4)\right]$$

$$y(t) = \begin{cases} 40\sin(t), & (0 \le) t < 2 \\ 40\sin(t) + 25\cos(t-2), & 2 \le t < 4 \\ 40\sin(t) + 25\cos(t-2) - 4t + 5\sin(t-4), & t \ge 4. \end{cases}$$

**Example 10.** *Find* $\mathcal{L}\{f(t)\}$*, where*

$$f(t) = \begin{cases} 5, & \text{if } 0 \le t < 1 \\ 3t, & \text{if } t \ge 1 \end{cases}.$$

$$f(t) = 5 + u(t-1)\left[3t - 5\right]$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{5}{s} + \mathcal{L}\left\{3u(t-1)\left[t-1\right] - 2u(t-1)\right\}$$

$$= \frac{5}{s} + 3e^{-s}\left(\frac{1}{s^2}\right) - 2\frac{e^{-s}}{s}$$

$$= \frac{5}{s} + \frac{3e^{-s}}{s^2} - \frac{2e^{-s}}{s}$$

## 8.7 Using Laplace Transforms to Solve DEs with Discontinuous Forcing Terms

Some DEs have forcing terms $f(t)$ with discontinuities/jumps at particular values of the independent variable. In general, these DEs have solutions that are **n**ot **d**ifferentiable at those points. The theory underlying this must be a little different to deal with this, but ultimately we require that solutions for these DEs are:

- **d**ifferentiable on each open interval where the forcing term is continuous; and
- **c**ontinuous at each point at which there is a discontinuity in the forcing term.

It is important that discontinuous forcing terms are written in terms of **H**eaviside functions if they are not already, so that we can take the Laplace transform easily.

**Example 11.** *Find the solution to the IVP*

$$y'' + 4y = \begin{cases} 0, & \text{if } 0 \le t < \pi \\ 1, & \text{if } \pi \le t < 2\pi \\ 0, & \text{if } t \ge 2\pi \end{cases}$$

$$y(0) = 1$$

$$y'(0) = 0.$$

$0 + u(t-\pi)[1-0]$
$+ u(t-2\pi)[0-1]$

First, convert to step functions:

$$y'' + 4y = u(t-\pi) - u(t-2\pi)$$

Take Laplace Transforms:

$$s^2\mathcal{L}\{y\} - s\overset{=1}{y(0)} - \overset{=0}{y'(0)} + 4\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\mathcal{L}\{y\}\left[s^2+4\right] = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} + s$$

$$\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-2\pi s}}{s(s^2+4)} + \frac{s}{s^2+4}$$

Pull exponentials out front of their terms; then we do partial fractions.

$$\mathcal{L}\{y\} = e^{-\pi s}\frac{1}{s(s^2+4)} - e^{-2\pi s}\frac{1}{s(s^2+4)} + \frac{s}{s^2+4}$$

Partial Fractions:

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

HSCU: $\boxed{A = 1/4}$

→ And common denoms:

$$\frac{1}{s(s^2+4)} = \frac{A(s^2+4) + (Bs+C)s}{s(s^2+4)}$$

Coeff of $s^2$: $0 = A + B \longrightarrow \boxed{B = -1/4}$

Coeff of $s$: $\boxed{0 = C}$

Thus we have

$$\frac{1}{s(s^2+4)} = \frac{1/4}{s} + \frac{-1/4\, s}{s^2+4}$$

(More space if needed...)

Sub back in to what we had:

$$\mathcal{L}\{y\} = e^{-\pi s}\left[\frac{1/4}{s} + \frac{-1/4\, s}{s^2+4}\right] - e^{-2\pi s}\left[\frac{1/4}{s} + \frac{-1/4\, s}{s^2+4}\right] + \frac{s}{s^2+4}$$

$$\mathcal{L}\{y\} = \frac{1}{4}\frac{e^{-\pi s}}{s} - \frac{1}{4}e^{-\pi s}\frac{s}{s^2+4} - \frac{1}{4}\frac{e^{-2\pi s}}{s} + \frac{1}{4}e^{-2\pi s}\frac{s}{s^2+4} + \frac{s}{s^2+4}$$

Inverse Laplace:

$$y = \frac{1}{4}u(t-\pi) - \frac{1}{4}u(t-\pi)\cos(2(t-\pi)) - \frac{1}{4}u(t-2\pi) + \frac{1}{4}u(t-2\pi)\cos(2(t-2\pi))$$

$$+\cos(2t).$$

~~$-\frac{1}{4}e^{-2\pi s}\cos(2(t-2\pi))$~~

Factor out and order step functions:

$$y = \cos(2t) + u(t-\pi)\left[\frac{1}{4} - \frac{1}{4}\cos(2(t-\pi))\right] + u(t-2\pi)\left[-\frac{1}{4} + \frac{1}{4}\cos(2(t-2\pi))\right]$$

Note that in this example, I could realize that the period of $\cos(2t)$ is $\pi$, so $\cos(2t) = \cos(2(t-\pi)) = \cos(2(t-2\pi))$. So I can rewrite the solution like this:

$$y = \cos(2t) + u(t-\pi)\left[\frac{1}{4} - \frac{1}{4}\cos(2t)\right] + u(t-2\pi)\left[-\frac{1}{4} + \frac{1}{4}\cos(2t)\right].$$