1.7 Solutions to DEs

Finally, after all of this setup, we're ready to talk about solving these DEs. But what does "solving" mean exactly?

Consider the equation $2x + 3 = 5$. Using grade-school algebra, we would use basic arithmetic to solve the equation.

$$2x = 2$$

$$\boxed{x = 1}$$

The solution to this equation is a **n** umber which, when substituted in for x, makes the left side equal to the right side.

T he **s** ame **t** hing **g** oes for differential equations! The only difference is that, rather than a **n** umber, our solutions will typically be **f** unctions.

So, if we are able to find a **f** unction that we can substitute in for the dependent variable in the DE so that left side is equal to right side, then that **f** unction is a solution to the DE!

Consider the second-order linear ODE

$$y''(t) + y(t) = 0$$

For this equation, it happens to be the case that $y(t) = \cos(t)$ is a solution. To see this, let's substitute this solution into the DE. To substitute into the y'' part, take the necessary derivatives of the solution:

$y(t) = \cos(t)$

$\rightarrow y'(t) = -\sin(t)$

$\rightarrow y''(t) = -\cos(t)$

Subbing in,

$LS = (-\cos(t)) + (\cos(t)) = 0 = RS$ ✓

For this DE, $y(t) = \cos(t)$ is not the only solution: $y(t) = \sin(t)$ is a solution too. So is any constant times $\sin(t)$ or $\cos(t)$, or any combination of these. So that means that $y(t) = 2\sin(t) + 3\cos(t)$ and $y(t) = \underline{4}\ \sin(t) + \underline{5}\ \cos(t)$ are both also solutions! You can verify them all (**For You to Try**).

But how can this be?

In fact, the **general solution** of this DE is

$$y(t) = C_1 \cos(t) + C_2 \sin(t).$$

where C_1 and C_2 are **a** ny **c** onstants **a** t **a** ll ! The general solution of a DE encompasses all possible solutions; in this case, there are an **i** nfinite number of them! **A** ny choice of C_1 and C_2 in the general solution gives us a solution to the DE. General solutions of DEs almost always include an infinite number of solutions that include constants like the ones in this example, which are often called **a** rbitrary **c** onstants .

✍ The function $y(t)$ might be a "solution," but what does that actually mean? $y(t)$ can represent all sorts of different things, depending on the equation: It could be a population that changes over time; a mass of a substance as it dissolves over time; the position of a moving object as time goes on, or many other things. By solving the DE and finding $y(t)$, you are actually determining what the behaviour of that quantity is, whatever it might represent.

1.8 Initial Value Problems (IVPs)

An **Initial Value Problem** is a **d** ifferential **e** quation that is paired with a set of **i** nitial **c** onditions. The effect of adding these **c** onditions is to narrow down the general solution, encompassing an **i** nfinite number of possible solutions, to just **o** ne. This gives a solution that can be represented by a single solution curve, sometimes called a solution **t** rajectory.

Consider the IVP:

$$\begin{aligned} y''(t) + y(t) &= 0 \\ y(0) &= 2 \\ y'(0) &= 0 \end{aligned}$$

The general solution to the DE, as we saw before, is $y(t) = C_1 \cos(t) + C_2 \sin(t)$ (though we still don't know how to actually go about finding this ourselves!). Now, we can use the initial conditions to narrow down this general solution to just a single solution.

First, we can use the $y(0)$ condition:

$$y(0) = 2 = C_1\cos(0) + \cancel{C_2\sin(0)}$$

$$\boxed{2 = C_1}$$

To deal with the second condition, take the derivative of the solution first:

$$y'(t) = C_1(-\sin(t)) + C_2\cos(t)$$

Then, substitute in as we did with the first condition.

$$y'(0) = 0 = \cancel{-C_1\sin(0)} + C_2\underbrace{\cos(0)}_{=1}.$$

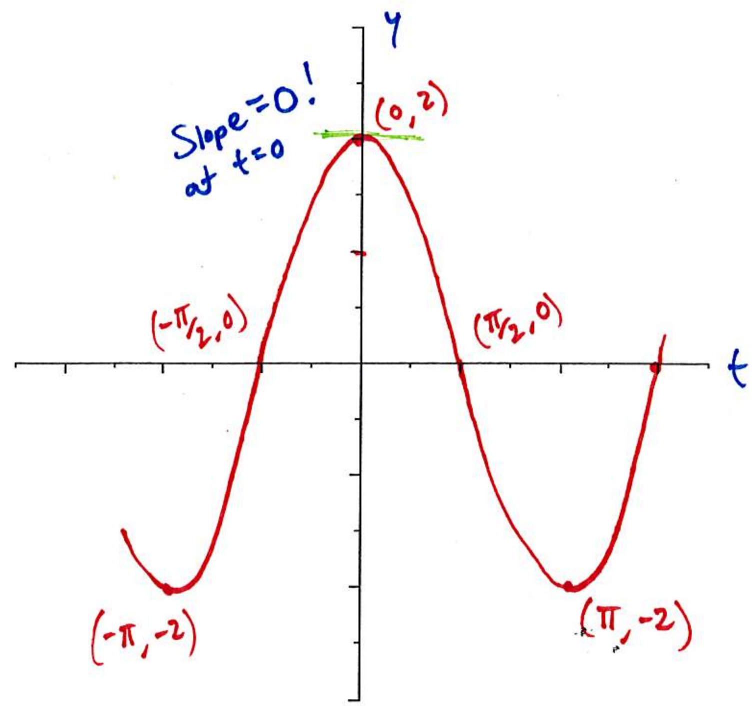
$$\boxed{C_2 = 0}$$

Thus, the solution to our IVP is:

$$y(t) = 2\cos(t) + \cancel{0\sin(t)}$$

$$\Rightarrow \boxed{y(t) = 2\cos(t)}$$

Let's quickly sketch the graph of $y(t) = 2\cos(t)$:



We can see that this function is very special because not only does it **s** <u>atisfy</u> the DE, but it **a** <u>lso</u> has a **h** <u>eight</u> of 2 and a **s** <u>lope</u> of 0 at $t = 0$.

In other words, it is the unique curve that fits the **i** <u>nitial</u> **c** <u>onditions</u> given in the IVP!

1.9 Existence and Uniqueness of Solutions to IVPs

Before we jump into trying to solve DEs and IVPs, we should be concerned with whether there are even solutions to be found **i**n **t**he **f**irst **p**lace. !

That is, given a particular IVP, does a solution **e**xist ? There is no guarantee that every equation has a solution to begin with.

As we saw in the example in the last section, specifying an initial condition narrows down the infinite number of solutions in the general solution to a **s**ingle solution curve. But how do we know that we will always get a **s**ingle answer? This is the question of **u**niqueness .

We will state (without proof - this requires some heavier math!) an **Existence and Uniqueness theorem** for linear, n^{th}-order ODEs.

Theorem.

Consider the n^{th}-order linear equation:

$$y^{(n)}(t)+f_{n-1}(t)y^{(n-1)}(t)\ldots+f_2(t)y''(t)+f_1(t)y'(t)+f_0(t)y(t)=g(t),$$

with n initial conditions

"Starting Time"

These are numbers.

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \\ &\;\;\cdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)} \\ &\cancel{y^{(n)}(t_0) = y_0^{(n)}.} \end{aligned}$$

where $y_0, y_0', \ldots, y_0^{(n)}$ are constants. If the functions $f_0, f_1, \ldots, f_{n-1}$ and g are continuous functions of t on an open interval (a,b), then there exists exactly one (unique) solution $y=\varphi(t)$ of the DE that also satisfies the set of initial conditions; moreover the solution exists throughout the interval (a,b).

So, existence and uniqueness depend entirely on the **c**ontinuity of the different functions found in a given DE!

✍ A consequence of this theorem is that for first-order equations, it is guaranteed that **o**nly **o**ne solution curve can possibly pass through any given point (t, y). In other words, solution curves for first-order equations **c**annot **c**ross one another when graphed. This will come in handy soon!

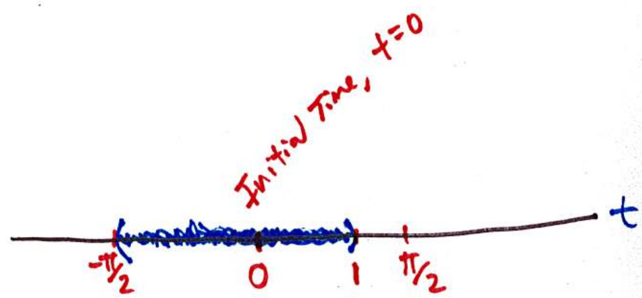
✍ While this existence and uniqueness theorem guarantees unique solutions for linear equations only, this does not mean that nonlinear equations cannot have unique solutions too; rather, they *may* not, and we may still be able to find solutions to particular problems. In fact, there are more existence and uniqueness theorems that deal specifically with nonlinear ODEs, at the expense of requiring additional more complicated hypotheses.

Example 5. *Determine the largest interval on which a unique solution is guaranteed to exist for the IVP:*

$$\begin{aligned} x'' - \tan(t)x' + t^3 x &= \frac{1}{t-1} \\ x(0) &= 1 \\ x'(0) &= 500 \end{aligned}$$

To answer this, first look for when "time starts" (here, the initial time is $t =$ $\underline{0}$). Then search for the interval of t, that includes that start time, on which *all* the functions involved in the DE are continuous! Here:

- $\tan(t)$ is continuous on $\underline{(-\pi/2, \pi/2)}$;
- t^3 is continuous on $\underline{(-\infty, \infty)}$;
- $\dfrac{1}{t-1}$ is continuous on $\underline{(-\infty, 1)}$.



Considering all three, a unique solution to this IVP is guaranteed to exist on $\underline{(-\pi/2, 1)}$.

Example 6. *Determine the interval on which a unique solution is guaranteed to exist for the IVP:*

$$(x^2 + 3x + 2)\frac{du}{dx} + u = 0$$
$$u(4) = 1$$

Put DE in standard form:

$$\frac{du}{dx} + \frac{1}{x^2+3x+2}u = 0$$

Check continuity of functions:

$\frac{1}{x^2+3x+2}$ has discontinuities wherever bottom = 0.

$$x^2+3x+2=0$$
$$(x+2)(x+1)=0$$
$$\rightarrow x=-2 \quad x=-1.$$

Pair this with our starting "time" $(x=4)$

$\therefore$ The interval on which a unique solution exists is $x \in (-1, \infty)$.

2 Solutions to First-Order DEs

2.1 Autonomous DEs and Direction Fields

Sometimes, ODEs are in a form that makes it very easy to determine the **q**ualitative **b**ehaviour of solutions without having to actually do any work to solve anything.

Especially easy to work with are **a**utonomous equations, which have the form:

$$\frac{dy}{dt} = f(y),$$

where the rate of change of y depends only upon y itself.

From the last section, we remember that as long as the DE satisfies the requirements for existence and uniqueness, we know that the different solutions composing the general solution of the DE can **n**ever **c**ross, since it is first-order. We can use this to aid in drawing a graph called a **d**irection **f**ield.

Helpful to us while drawing such graphs are:

- Lines along which $\dfrac{dy}{dt} = 0$. This tells us where solutions have **h** orizontal ____ **t** angent ____ .

- Areas where $\dfrac{dy}{dt} < 0$ (or > 0). This tells us when y values are **d** ecreasing ____ (or **i** ncreasing ____) with increasing t.

An **e** quilibrium ____ **v** alue ____ for a DE is a value of y (a height!) such that if a solution starts at that height at the initial t-value, t_0, it stays there for all $t > t_0$. For autonomous equations, these can always be found by setting $\dfrac{dy}{dt}$ to **z** ero ____ and solving for y.

Example 1. *Find the equilibria for the DE given by*

$$y'(t) = y(t) - 3.$$

Draw a direction field for the DE and draw in some representative solution curves.

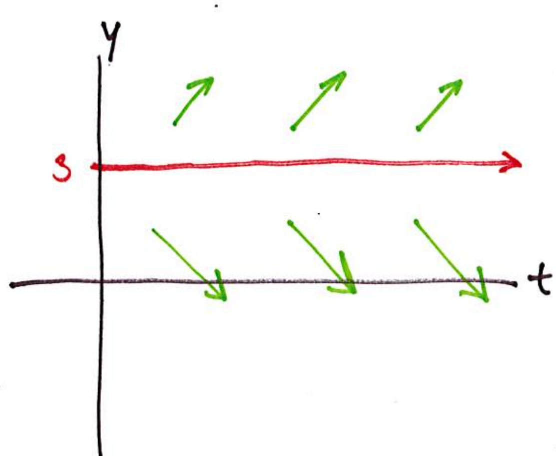
Finding the equilibrium value is simple! The DE is autonomous, so set $y'(t)$ to zero and solve for y:

$$0 = y - 3$$

$$\boxed{y = 3}$$

Now, use some calculus muscle! We can see that:

- For $y >$ 3 , $y'(t)$ is **p**ositive ; hence solution curves are **i**ncreasing .
- For $y <$ 3 , $y'(t)$ is **n**egative , so solution curves are **d**ecreasing .



How can we get the **c**urvature of the different lines correct?

This, of course, relates to **c**oncavity, which is related to the **s**econd **d**erivative. If we consider autonomous equations only for the moment, then if $\frac{dy}{dt} = f(y)$, we can take the derivative of both sides *with respect to t* by carefully using the chain rule:

$$\frac{d^2y}{dt^2} = \underbrace{f'(y)}_{\text{"}df/dy\text{"}}\frac{dy}{dt}$$

If $f'(y)$ is **n**egative, then the second derivative $\left(\frac{d^2y}{dt^2}\right)$ will be the **o**pposite sign of the first derivative $\left(\frac{dy}{dt}\right)$. This corresponds to *slowing growth or decline.*

or

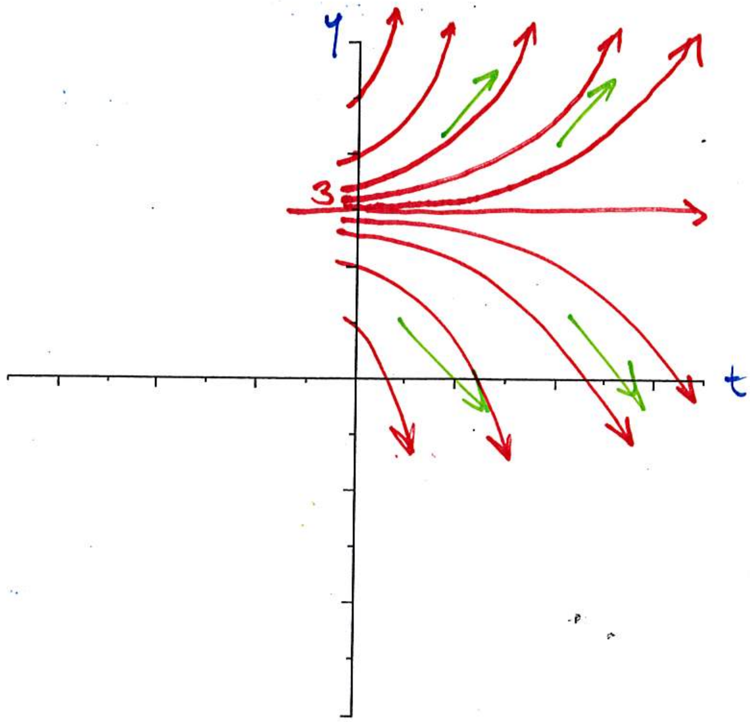
If $f'(y)$ is **p**ositive, then the second and first derivatives will be the ~~sign~~ same sign, corresponding to *quickening growth or decline.*

or

In our example, $f(y) = y - 3$, which means that $f'(y) =$ 1.

This is positive everywhere, so we obtain quickening growth wherever the graph is increasing, and quickening decline wherever the graph is decreasing.

Putting everything together, we (finally!) come up with our direction field:



✍ In this example, $y = 3$ is called an **unstable** equilibrium point because nearby solution trajectories are repelled from it as time goes forward. An equilibrium point that attracts nearby solutions (i.e. they become infinitely close as $t \to \infty$) is often called **asymptotically stable**. There are other kinds of stability too!

✍ "What solutions do as $t \to \infty$" is an important notion that will be revisited in this course and in others. Be prepared to answer this question, either by carefully analyzing a graph or by taking a limit!

Example 2. *Find the equilibria for the DE given by*

$$\frac{dy}{dt} = y^2 - 5y + 6. \quad = (y-2)(y-3)$$

Draw a direction field for the DE and draw in some representative solution curves.

Equilibria are found by setting $\frac{dy}{dt} = 0 \rightarrow y^2 - 5y + 6 = 0$

$(y-2)(y-3) = 0$

$\hookrightarrow y = 2, \quad y = 3$ are both equilibrium values

Also, $\frac{dy}{dt}$ is positive if: $y > 3$ or $y < 2$

and negative if: $2 < y < 3$.

Now, curvature:

I need $f'(y)$.

$f'(y) = 2y - 5$

We see that $2y - 5$

is positive for: $y > 5/2$ → Quickening Growth/Decline

and negative for: $y < 5/2$ → Slowing Growth/Decline.