5 Some Basic Solution Techniques for Higher-Order Linear DEs

5.1 The Method of Reduction of Order

Suppose we already somehow know one solution, $y_1(x)$, to the second-order linear DE:

$$L[y] = y'' + p(x)y' + q(x)y = 0,$$

We can always find a second solution, **l** inearly **i** ndependent to the first, by assuming the second solution is of the form $y_2(x) = v(x)y_1(x)$, where $v(x)$ is some unknown function of the independent variable.

This is a pretty big claim! In fact, we could prove it in general, leading to a (rather complicated) formula. Give it a whirl if you like (**For You to Try**), but we will see how it works through an example.

Example 4. *Consider the DE given by*

$$t^2y'' + 5ty' - 12y = 0.$$

Given that $y_1(t) = t^2$ is one solution of this equation, find another linearly independent solution and hence the general solution to the DE.

Let $y_2(t) = v(t)\,y_1(t) = v(t)\,t^2$, v is an unknown function.

Assuming y_2 is itself a solution, take derivatives and sub in.

$$y_2' = v't^2 + 2vt$$

$$y_2'' = v''t^2 + 2v't + 2v't + 2v = v''t^2 + 4v't + 2v$$

Sub into the DE:

$$t^2(v''t^2 + 4v't + 2v) + 5t(v't^2 + 2vt) - 12vt^2 = 0$$

Sort by derivatives of v:

$$v''(t^4) + v'(4t^3 + 5t^3) + \cancel{v(2t^2 + 10t^2 - 12t^2)} = 0$$

The coeff of v should always work out to zero!

This is a first-order DE in v'.

Do a substitution:

$$\text{Let } u = v', \text{ so } u' = v''$$

We obtain:

$$t^4u' + 9t^3u = 0$$

$$\rightarrow \quad u' + \frac{9}{t}u = 0$$

Integrating Factor: $\mu = e^{\int \frac{9}{t}\,dt} = e^{9\ln|t| + C} = t^9$

$$\rightarrow \quad t^9u' + 9t^8u = 0$$

$$\rightarrow \quad (t^9u)' = 0$$

$$\rightarrow \quad t^9u = C$$

$$u = Ct^{-9}$$

$$\rightarrow \quad v' = Ct^{-9}$$

$$\rightarrow \quad v = \frac{Ct^{-8}}{-8} + D$$

Just another arbitrary constant.

Our second solution is

$$y_2(t) = (Ct^{-8} + D)\underbrace{t^2}_{y_1}$$

$$= \underbrace{Ct^{-6}}_{\text{New linearly indep. Sol'n.}} + \underbrace{Dt^2}_{\text{Our old solution } y_1.}$$

5.2 Solving Homogeneous Linear Second-Order DEs with Constant Coefficients

Consider the DE with the standard form

$$ay''(x) + by'(x) + cy(x) = 0,$$

where $a \neq 0$, b, and c are all constants. In search of solutions to this equation, observe what happens if we try to find one of the form $y = e^{rx}$. The derivatives of y are:

$y' =$ re^{rx} $\qquad\qquad y'' =$ r^2e^{rx}

and substituting these into the DE gives:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

We can factor out the exponential to get:

$$e^{rx}(ar^2 + br + c) = 0$$

... and since $e^{rx} \neq 0$ ever, this equation is satisfied only if:

$$ar^2 + br + c = 0$$

This looks like a **p**olynomial that we can solve for r! We call this the **characteristic equation** or **auxiliary equation** of the DE.

Solving the characteristic polynomial for r will yield two roots; these two values of r give us two linearly independent solutions to the DE. Note that whether we can solve by factoring this polynomial or not, we can always use the **q** uadratic **f** ormula to find r, yielding:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives two values for r, which we will call r_1 and r_2. We obtain different cases depending on whether the values of r_1 and r_2 we obtain are **r** eal, **r** epeated, or **c** omplex.

Case 1: r_1 and r_2 are real roots, with $r_1 \neq r_2$

In this case, $b^2 - 4ac \; \boxed{>} \; 0$. The general solution to the DE is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 5. *Find the general solution to*

$$x''(t) - 3x'(t) - 4x(t) = 0.$$

Characteristic Eq:

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$r = 4, \quad r = -1$$

General Solution: $\boxed{x(t) = C_1 e^{4t} + C_2 e^{-t}}$

Case 2: r_1 and r_2 are real roots, with $r_1 = r_2$

In this case, $b^2 - 4ac \boxed{=} 0$. This gives a single repeated root of $r_1 = -\dfrac{b}{2a}$ and thus only one solution, $y_1 = e^{-\frac{b}{2a}x}$. We know, however, that we need two linearly independent solutions for a second-order equation! Luckily, we've already covered a technique that we can use to find a second solution if we know just one: **R**eduction **o**f **O**rder ! Using this leads to a general solution of

$$
\begin{aligned}
y(x) &= c_1 e^{r_1 x} + c_2 x e^{r_1 x}, \quad \text{or} \\
y(x) &= (c_1 + c_2 x) e^{r_1 x}
\end{aligned}
$$

✍ Showing that this is true by using Reduction of Order is actually really cool, and we'll leave it **For You to Try**: Good luck!

Example 6. *Find the general solution to*

$$4y''(t) + 12y'(t) + 9y(t) = 0.$$

Characteristic Eq: $4r^2 + 12r + 9 = 0$

$$(2r+3)^2 = 0$$

$$\rightarrow \quad r = -3/2 \quad , \text{ repeated!}$$

$$\boxed{y(t) = C_1 e^{-3/2 t} + C_2 t e^{-3/2 t}}$$

Case 3: r_1 and r_2 are a complex pair

In this case, $b^2 - 4ac$ $\boxed{<}$ 0, and the roots will be in the form of

$$r = \alpha \pm j\beta,$$

where α and β are real numbers, with $j = \sqrt{-1}$. Following the example of Case 1, we would obtain a general solution of

$$y = c_1 e^{(\alpha + j\beta)x} + c_2 e^{(\alpha - j\beta)x}.$$

But, this is a **c**omplex solution, and we are interested only in **r**eal -**v**alued solutions to our DE. How should we proceed?

Recall Euler's formula:

$$e^{jx} = \cos(x) + j\sin(x)$$

We can use this formula to rewrite our exponentials! Specifically,

$$e^{(\alpha+j\beta)x} = e^{\alpha x}e^{j\beta x} = e^{\alpha x}\left[\cos(\beta x) + j\sin(\beta x)\right], \text{ and}$$

$$e^{(\alpha-j\beta)x} = e^{\alpha x}e^{j(-\beta x)} = e^{\alpha x}\left[\cos(-\beta x) + j\sin(-\beta x)\right]$$

$$= e^{\alpha x}\left[\cos(\beta x) - j\sin(\beta x)\right]$$

Cos is even! Sin is odd!

Subbing these into the complex general solution we had, we get:

$$y = C_1 e^{\alpha x}\left(\cos(\beta x) + j\sin(\beta x)\right) + C_2 e^{\alpha x}\left(\cos(\beta x) - j\sin(\beta x)\right).$$

We can group the sin and cos functions together to obtain:

$$y = e^{\alpha x}\left[\cos(\beta x)\underbrace{(C_1 + C_2)}_{\text{Another arbitrary constant}} + \sin(\beta x)\underbrace{(C_1 - C_2)j}_{\text{Another arbitrary constant.}}\right]$$

Thus, we have here a linear combination of two different functions that are linearly independent from one another: $e^{\alpha x}\cos(\beta x)$, and $e^{\alpha x}\sin(\beta x)$, which both serve as solutions to the DE. Using our superposition principles, then, the general solution for this case is given by:

$$e^{\alpha x}\left[C_1\cos(\beta x) + C_2\sin(\beta x)\right].$$

Example 7. *Find the general solution to*

$$y''(t) + y'(t) + 4y(t) = 0$$
$$y(0) = 0$$
$$y'(0) = 1$$

Characteristic Eq:

$$r^2 + r + 4 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(4)}}{2} = \underbrace{-\frac{1}{2}}_{\text{Real Part "}\alpha\text{"}} \pm \underbrace{\frac{\sqrt{15}}{2}}_{\text{Imaginary Part "}\beta\text{"}} j$$

$$\text{So, } y(t) = e^{-\frac{1}{2}t}\left[C_1 \cos\left(\frac{\sqrt{15}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{15}}{2}t\right)\right]$$

Apply the ICs:

$$y(0)=0 \longrightarrow 0 = (1)\left[C_1(1) + \cancel{C_2(0)}\right] \longrightarrow \boxed{C_1 = 0}$$

We know C_1 is zero - plugging this in before taking the derivative for the second IC makes life easier!

$$\text{ie. } y(t) = e^{-\frac{1}{2}t}\left[C_2 \sin\left(\frac{\sqrt{15}}{2}t\right)\right]$$

$$\rightarrow y'(t) = C_2\left[-\frac{1}{2}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right) + e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right)\left(\frac{\sqrt{15}}{2}\right)\right]$$

$$y'(0)=1 \longrightarrow 1 = C_2\left[\cancel{-\frac{1}{2}(1)(0)} + (1)(1)\left(\frac{\sqrt{15}}{2}\right)\right]$$

$$\hookrightarrow \boxed{C_2 = \frac{2}{\sqrt{15}}}$$

We obtain the solution to the IVP:

$$\boxed{y(t) = \frac{2}{\sqrt{15}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)}$$

6 Solution Methods for Nonhomogeneous DEs

6.1 Some Preliminary Theory and Definitions

Now, let's deal with the case where we have a DE of the form:

$$L[y] = a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = g(x)$$

That is, the case where there isn't just a **z**ero on the right-hand side. It turns out that this simple twist makes solving a whole lot more involved. $g(x)$ is often called the **f**orcing **t**erm of the DE. Also, we will often call

$$L[y] = 0$$

...that is, the same DE but with a zero on the right-hand side... the **c**orresponding **h**omogeneous **e**quation. .

To see what is involved in the solving one of these DEs, first consider two hypothetical solutions, which we will call Y_1 and Y_2.

If Y_1 and Y_2 are solutions to $L[y] = g(x)$, then we must have that:

$$L[Y_1] = \underline{g(x)} \qquad \text{and} \qquad L[Y_2] = \underline{g(x)}$$

So then it follows that

$$L[Y_1] - L[Y_2] = \underline{g(x)} - \underline{g(x)} = \underline{0}.$$

But, L is a **l**inear **o**perator, so

$$L[Y_1] - L[Y_2] = \underline{L[Y_1 - Y_2]} = 0$$

That is, $Y_1 - Y_2$ is a solution to the *corresponding homogeneous equation* $L[y] = 0$.

Luckily, we already know how to solve homogeneous equations, from the previous section! We know that an n^{th}-order equation must have a fundamental solution set $\{y_1, \ldots, y_n\}$ and a general solution that consists of a linear combination of all of these linearly independent solutions:

$$y = c_1 y_1 + \ldots + c_n y_n$$

Thus, we can say that for *our* hypothetical solution $Y_1 - Y_2$,

$$Y_1 - Y_2 = c_1 y_1 + \ldots + c_n y_n.$$

or rearranging,

$$Y_1 = \underbrace{c_1 y_1 + \ldots + c_n y_n}_{\text{general solution to the homogeneous problem.}} + \overbrace{Y_2}^{\text{a particular solution}}$$

We find that any solution Y_1 has two components:

- The general solution $y = c_1 y_1 + \ldots + c_n y_n$ of the corresponding homogeneous equation (we denote this $\boxed{\mathbf{y_h}}$), containing all of the **a**rbitrary **c**onstants ;
- A "**p**articular" solution that itself satisfies $L[y] = g(x)$ (we denote this $\mathbf{y_p}$). This part contains **n**o **a**rbitrary **c**onstants .

Finding y_h should be straightforward. So our problem now becomes: How do we find a particular solution y_p?

6.2 The Method of Undetermined Coefficients

This method is a bit of a **c**ommon **s**ense approach to figuring out what y_p should be. Given a DE

$$a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = g(x),$$

the idea is to make an **e**ducated **g**uess as to what we would need to substitute in the left-hand side in order to make the left side equal to the right side. Of course, that will depend heavily upon what functions make up $g(x)$!

For example, we would expect that:

- if $g(x)$ is an exponential function, then y_p must be an exponential function as well, since the derivative of an exponential is an **e**xponential ;
- if $g(x)$ is a sin or cos function, it stands to reason that y_p has to be made up of sin and/or cos functions.
- if $g(x)$ is polynomial of degree n, then y_p should also be, since derivatives of powers give **m**ore **p**owers .

These types of functions, and combinations of them, are the kinds of functions that this method works well for! Let's start with a basic example:

Example 1. *Find the general solution to*

$$\underbrace{y'' + 6y' + 9y}_{\ddot\smile} = \underbrace{2e^{-x} + \sin(2x) + 3x^2}_{}.$$

First, find y_h, the solution to $y'' + 6y' + 9y = 0$.

Char Eq: $r^2 + 6r + 9 = 0 \rightarrow (r+3)^2 = 0$

$\rightarrow r = -3$, repeated.

So, $\boxed{y_h = C_1 e^{-3x} + C_2 x e^{-3x}}$

Now, find y_p!

Assume a form of:

$$y_p = A_1 e^{-x} + A_2 \cos(2x) + A_3 \sin(2x) + A_4 x^2 + A_5 x + A_6$$

We need to determine the "undetermined" coefficients $A_1, A_2, A_3, A_4, A_5, A_6$ so that LS = RS in the original DE. ie. these constants are NOT arbitrary.

Sub into the DE: Yeah, we've got to take two derivatives!

$$y_p' = -A_1 e^{-x} - 2A_2 \sin(2x) + 2A_3 \cos(2x) + 2A_4 x + A_5$$

$$y_p'' = A_1 e^{-x} - 4A_2 \cos(2x) - 4A_3 \sin(2x) + 2A_4$$

Then, subbing in:

$$A_1 e^{-x} - 4A_2 \cos(2x) - 4A_3 \sin(2x) + 2A_4$$
$$+6\left[-A_1 e^{-x} - 2A_2 \sin(2x) + 2A_3 \cos(2x) + 2A_4 x + A_5\right]$$
$$+9\left[A_1 e^{-x} + A_2 \cos(2x) + A_3 \sin(2x) + A_4 x^2 + A_5 x + A_6\right]$$
$$= 2e^{-x} + \sin(2x) + 3x^2$$

Here's some more space...

Group by Function Type:

$$e^{-x}\left[A_1 - 6A_1 + 9A_1\right] + \cos(2x)\left[-4A_2 + 12A_3 + 9A_2\right]$$

$$+ \sin(2x)\left[-4A_3 - 12A_2 + 9A_3\right] + x^2\left[9A_4\right] + x\left[12A_4 + 9A_5\right] + \left[2A_4 + 6A_5 + 9A_6\right]$$

Match Coefficients to get a system of equations: $= 2e^{-x} + \sin(2x) + 3x^2$

$4A_1 = 2$ (coeff of e^{-x}) $\rightarrow \boxed{A_1 = 1/2}$

① $5A_2 + 12A_3 = 0$ (coeff of $\cos(2x)$)

② $-12A_2 + 5A_3 = 1$ (coeff of $\sin(2x)$)

$9A_4 = 3$ (coeff of x^2) $\rightarrow \boxed{A_4 = 1/3}$

$12A_4 + 9A_5 = 0$ (coeff of x) $\rightarrow 12(1/3) + 9A_5 = 0$

$\boxed{A_5 = -4/9}$

$2A_4 + 6A_5 + 9A_6 = 0$ (constant terms) $\rightarrow 2(\frac{1}{3}) + 6(-4/9) + 9A_6 = 0 \rightarrow \boxed{A_6 = 2/9}$

Now solve ① and ②.

From ① $A_2 = \dfrac{-12A_3}{5}$

Sub into ② $\rightarrow -12\left(-\frac{12}{5}\right)A_3 + 5A_3 = 1$

$\rightarrow \frac{144}{5}A_3 + 5A_3 = 1$

$\rightarrow \boxed{A_3 = \frac{5}{169}} \longrightarrow \boxed{A_2 = \frac{-12}{169}}$

✍ Sometimes there are a few wrinkles with this method that complicate matters. More details on the next page!

So, $y_p = \frac{1}{2}e^{-x} - \frac{12}{169}\cos(2x) + \frac{5}{169}\sin(2x) + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{2}{9}$.

General Solution: $y(x) =$ (Our y_h from before) + (the y_p we just found).