As we progress through the course, we will find that strategies for finding the solutions for ODEs are highly dependent on what particular form the equations take. For now, we'll focus on methods of finding the solutions to first-order ODEs.

### 2.2 "Not-Really DEs"

The simplest type of ODE that we can solve are those that don't require any special techniques at all outside of **i**ntegration .

These ODEs are always of the form:

$$y'(t) = g(t).$$

To solve for the function $y$, we need only **i**ntegrate **b**oth **s**ides with respect to the independent variable.

✍ DON'T forget your arbitrary constants that arise due to integration. They make up a crucial part of your solution!

**Example 3.** *Find the general solution to* $x'(t) = \frac{1}{\underbrace{(t-2)^3}_{\text{A function of only } t. \text{ Integrate both sides!}}}.$

$$x(t) = \int \frac{1}{(t-2)^3}\, dt$$

$$\boxed{= \frac{-1}{2(t-2)^2} + C}$$

**Example 4.** *Find the solution to the IVP given by:*

$$\frac{dx}{dt} = 1 - e^{-3t}$$

$$x(0) = 1$$

Integrate:

$$x(t) = \int 1 - e^{-3t}\, dt$$

$$= t + \frac{1}{3}e^{-3t} + C. \quad \text{(General Solution)}$$

Now apply the initial condition:

$$x(0) = 1 = 0 + \frac{1}{3}e^{-3(0)} + C$$

$$\rightarrow 1 = \frac{1}{3} + C$$

$$\rightarrow C = 2/3$$

Thus, the solution to the IVP is

$$\boxed{x(t) = t + \frac{1}{3}e^{-3t} + 2/3}$$

In the more general, $n^{th}$-order case, if we have a DE equation in the form of

$$y^{(n)}(t) = g(t),$$

then we can obtain the solution by integrating $n$ times.

**Example 5.** *A particle's acceleration is given by* $\dfrac{-2t}{(1-t^2)^{3/2}}$ $m/s^2$*. Suppose that the particle's initial velocity is* 1 $m/s$ *and set the initial position equal to* 0 $m$ *. Find an expression for the particle's position at time* $t$.

Let $x(t)$ be the position of the particle at time $t$.

We have $x''(t) = \dfrac{-2t}{(1-t^2)^{3/2}}$

$x(0) = 0$

$x'(0) = 1$

We can obtain the solution by integrating twice!

$$x'(t) = \int \frac{-2t}{(1-t^2)^{3/2}}\,dt$$

$$\rightarrow x'(t) = \frac{-2}{\sqrt{1-t^2}} + C$$

We can apply an initial condition here:

$$x'(0) = 1 = -2 + C$$

$$\rightarrow \boxed{C = 3}$$

Now we have

$$x'(t) = \frac{-2}{\sqrt{1-t^2}} + 3$$

$$x(t) = \int \left(\frac{-2}{\sqrt{1-t^2}} + 3\right) dt$$

$$= -2\arcsin(t) + 3t + D$$

Apply the other IC here:

$$x(0) = 0 = \cancel{-2\arcsin(0)} + \cancel{3(0)} + D .$$

$$\boxed{D = 0}$$

$\therefore$ the particle's position is $\boxed{x(t) = -2\arcsin(t) + 3t}$

### 2.3 Linear DEs and Integrating Factors

To demonstrate the idea of how we will go about solving first-order linear DEs, we will leap right into an example.

**Example 6.** *Consider the first-order linear DE*

$$t^2y' + 2ty = e^{2t}$$

Remember that the goal for solving this DE is to find a **f**unction that we can sub into $y(t)$ in order to satisfy the equation. Well, notice that if we were to let $a(t) = t^2$, the equation is in the form:

$$\underline{a(t)}\,\underline{y'} + \underline{a'(t)}\,\underline{y} = e^{2t}$$

This is significant, because we can use **p**roduct **r**ule **b**ackwards in order to rewrite this as:

$$(a(t)y)' = e^{2t}$$

Applying this notion to our example, we obtain

$$(t^2y)' = e^{2t}$$

We are now able to simply integrate both sides, to get

$$t^2 y = \frac{1}{2}e^{2t} + C$$

Finally, isolating for $y$, we obtain

$$\boxed{y = \frac{1}{2t^2}e^{2t} + \frac{C}{t^2}}$$

This is the general solution to the ODE.

Certainly, our equation must have been **p**retty **d**arn **s**pecial if it was set up so that we could use product rule backwards without a problem. What if we can't do this step?

Let's explore another example!

**Example 7.** *Consider the first-order linear ODE*

$$y' + 2y = 5.$$

Here, this equation isn't in a form that we can readily use the product rule backwards on. In cases like these, we search for an **i**ntegrating **f**actor $\mu(t)$ that we can multiply every term in the equation by in order to *get* it in a form that we can use. That is, we want to find $\mu(t)$ so that we obtain the equation

$$\mu(t)y' + 2\mu(t)y = 5\mu(t),$$

where we want $2\mu(t)$ (the coefficient in front of the $y$), to be the derivative of the $\mu(t)$ (the coefficient in front of the $y'$). This would set us up to be able to use product rule backwards like in the first example, while **p**reserving the original DE! To do this, we need to make sure that:

$$\frac{d}{dt}\mu(t) = 2\mu(t)$$

Assume that $\mu(t) > 0$ so that we can divide by it (we will soon see that this assumption is valid). We obtain

$$\frac{1}{\mu(t)}\frac{d}{dt}\mu(t) = 2.$$

We can rewrite this using derivative rules and then integrate both sides to solve for $\mu(t)$:

$$\frac{d}{dt}\left(\ln(\mu(t))\right) = 2.$$

$$\rightarrow \quad \ln(\mu(t)) = 2t + C.$$

$$\rightarrow \quad \mu(t) = e^{2t+C} = e^{2t}e^{C}$$ using exponent rules.

$$= Ce^{2t}$$

A "new" $C = e^{\text{old } C}$.

We have found an integrating factor! In fact, we have found a whole bunch of them (one for each value of $c$), and we only need one, so we have the **f**reedom to **c**hoose any one that works. Choose $c = 1$ to make the expression as simple as possible.

$$\boxed{\mu(t) = e^{2t}.}$$

We now multiply the DE by the integrating factor that we found in order to get:

$$e^{2t}y' + 2e^{2t}y = 5e^{2t}$$

Now we can see that the equation is in the form that we desire: $2e^{2t}$ is the derivative of $e^{2t}$, and so we can use the product rule backwards as we did in the first example! Follow the same process as we did for the first example in order to arrive at the general solution:

$$(e^{2t}y)' = 5e^{2t}$$

Integrate:

$$e^{2t}y = \frac{5}{2}e^{2t} + C$$

Isolate:

$$\boxed{y = \frac{5}{2} + Ce^{-2t}}$$

✍ In fact, using the approach that we detailed in the previous example, as long as we have an equation in the form:

$$y' + p(t)y = g(t),$$

using those same steps will always allow us to find an integrating factor $\mu(t)$, yielding a formula which we will use from now on:

$$\mu(t) = e^{\int p(t)\ dt}$$

**Example 8.** *Find the general solution to:*

$$\frac{dy}{dx} + x^4 y = e^{-x^5}$$

First-Order Linear!

To find an integrating factor, use our new formula!

$$\mu(x) = e^{\int x^4\,dx} = e^{\frac{x^5}{5}+c} = \underbrace{e^{c}}\,e^{x^5/5} = Ce^{x^5/5}$$

Choose $C=1$, to make things simple.

Multiply through:

$$\underbrace{e^{x^5/5}}\frac{dy}{dx} + \underbrace{x^4 e^{x^5/5}}\,y = e^{-x^5}e^{x^5/5}$$

Product Rule Backwards:

$$\left(e^{x^5/5}\,y\right)' = e^{-\frac{4}{5}x^5}$$

Integrate:

$$e^{x^5/5}\,y = \int e^{-\frac{4}{5}x^5}\,dx$$

Isolate for $y$:

$$y(x) = e^{-x^5/5}\int e^{-\frac{4}{5}x^5}\,dx.$$

## 2.4 Separable DEs

Product rule isn't the only calculus trick we can use to solve certain first-order DEs. Let's take a look at another type of equation. In this section, we will consider equations of the form:

$$f(y)\frac{dy}{dt} = g(t)$$

By using the **c**hain **r**ule **b**ackwards, we could actually rewrite this as

$$\frac{d}{dt}(F(y)) = g(t),$$

where $\frac{d}{dy}(F(y)) = f(y)$. **I**ntegrating both sides with respect to $t$, we obtain

$$F(y) = \int g(t)dt.$$

If $F(y)$ is such that we can isolate for the **d**ependent **v**ariable, we do so to obtain an explicit solution for $y(t)$.

This process involves an integration on the left side with respect to $y$ (when we find the antiderivative $F(y)$), and then an integration on the right side with respect to $t$ (in the final step). To make things seem simpler, the following notation is often used:

$$f(y)\frac{dy}{dt} = g(t) \qquad \longrightarrow \qquad f(y)\,dy = g(t)\,dt$$

This resembles "multiplying up dt" (though that is *not* what is happening). This "separates the variables" and each side may then be integrated with respect to its **o**wn **v**ariables:

$$\int f(y)\,dy = \int g(t)\,dt$$

✍ Don't forget to add an **a**rbitrary **c**onstant on one side when you integrate!

✍ Be very careful: whenever you divide by a quantity, you are making an assumption that this quantity is not **z**ero. By making this assumption, you may be discarding valid solutions! Often, simple constant solutions are missed this way.

We will start by carefully applying this method to a simple DE.

**Example 1.** *Find the general solution to the DE:*

$$y'(t) = 6y(t).$$

We first recognize that $y(t) =$ 0 , the zero function, is a simple constant solution to this DE. Putting that possibility aside for the moment, we now assume $y \neq 0$ to use the technique we just introduced. First, separate the variables by division:

$$\frac{1}{y}y' = 6$$

Now, integrate both sides and we obtain:

$$\ln|y| = 6t + C,$$

where $C$ is any real constant.

It appears as though we can isolate for $y$, so we can take the exponential of both sides to get:

$$|y| = e^{6t+C}. \quad = e^{c} e^{6t}$$

Since $e^C$ gives us just another constant, we can let $c = e^C$ be a new constant. The new $c$ is, of course, **p**ositive since $e^{\text{anything}}$ is always **p**ositive. We come to:

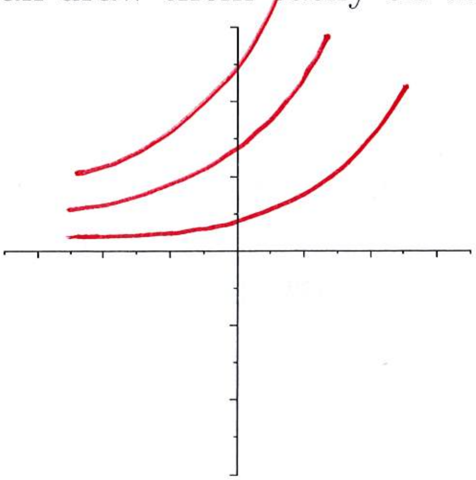
$$|y| = ce^{6t}.$$

The absolute values might seem annoying at first. But what did we first do to tackle absolute values way back in previous courses? We **d**id **c**ases !!

**Case 1:** $y > 0$. Then $|y| = y$. Our solution becomes $y = ce^{6t}$.

No matter what the values of $C$ are, these functions are always

positive ! We can draw them easily on the $t - y$ axis:

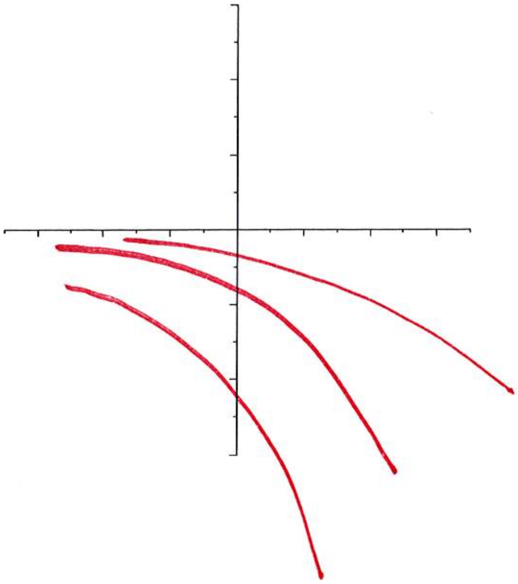


**Case 2:** $y < 0$. Then $|y| = -y$. Our solution becomes $-y = ce^{6t}$

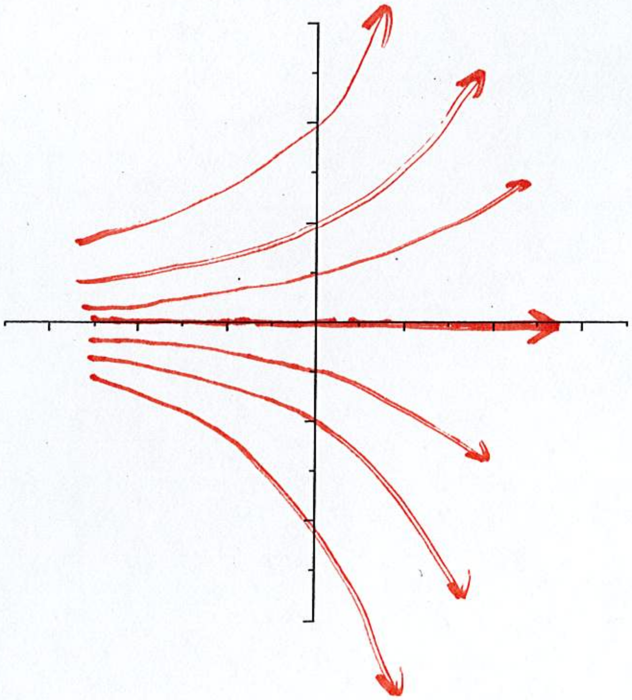
or $y = -ce^{6t}$

No matter what the values of $C$ are, these functions are always

negative ! We can plot these curves too:



Putting these graphs together, along with the very first solution we found ($y(t) = 0$), we obtain the family of curves given by $y = ce^t$, where $c$ is *any* real constant.



In practice, many people tend not to be quite so careful and/or thorough, though this kind of mathematical background and precision *should* be kept in mind. Typically, a more concise solution to the last example might look something like this:

$$y' = 6y$$

$$\frac{1}{y}y' = 6$$

$$\ln|y| = 6t + C$$

$$|y| = e^{6t+C} \longrightarrow y = Ce^{6t}$$

$^*$ $y = 0$ is a solution itself

↖ this new $C$ takes into account the possibility of negative signs due to abs. vals. and the zero solution we found before!

✍ For many nonlinear DEs, after integration, it will be impossible to rearrange to isolate for the dependent variable. In this case, it is okay for the solution to be defined only i mplicitly.

**Example 2.** *Find the general solution to the following DEs:*

** First, note that $x = 0$ is a solution.

a) $\frac{\sqrt{x}}{t^3} = tx'$

Divide!

$$\frac{1}{t^4} = \frac{1}{\sqrt{x}} x'$$

Integrate both sides!

$$-\frac{1}{3t^3} = 2\sqrt{x} + C$$

Isolate for $x$ if you can:

$$2\sqrt{x} = -\frac{1}{3t^3} + C$$

*Note, "$-C$" is the same as "$+C$"

$$\sqrt{x} = -\frac{1}{6t^3} + C$$

** Note that "$+\frac{C}{2}$" is the same as "$+C$"!

$$\boxed{x = \left(-\frac{1}{6t^3} + C\right)^2}$$

or

$$\boxed{x = 0}$$

(from before)

b) $\cos(x)y' - y\sin(x) = 0$

$$\cos(x)\, y' = y\sin(x)$$

→ *Note $y(x) = 0$ is a solution!

$$\frac{1}{y} y' = \frac{\sin(x)}{\cos(x)}$$

Integrate!

$$\ln|y| = -\ln|\cos(x)| + C.$$

$$e^{\ln|y|} = e^{-\ln|\cos(x)| + C}$$

$$e^{\ln|y|} = e^{\ln|\sec(x)| + C}$$

$$|y| = e^{C}|\sec(x)|$$

→ Redefine $C$ again to "catch" any negative signs arising from abs. values. *and the zero solution in the start.

$$\boxed{y = C\sec(x)}$$

**Example 3.** *Solve the IVP given by*

$$y\frac{dy}{dt} = te^{-y},$$

$$y(1) = 2$$

Separate variables!

$$ye^{y}\frac{dy}{dt} = t$$

Integrate both sides!

$$ye^{y} - e^{y} = \frac{1}{2}t^{2} + C$$

$$\int ye^{y}\,dy$$

$u = y$ $\quad \frac{dv}{dy} = e^{y}$

$\frac{du}{dy} = 1$ $\quad v = e^{y}$

$$= ye^{y} - \int e^{y}\,dy$$

$$= ye^{y} - e^{y} + C.$$

Use the initial condition right now!

"When t is 1, y is 2"

↳ Some wise man.

$$2e^{2} - e^{2} = \frac{1}{2}(1)^{2} + C$$

$$\boxed{e^{2} - \frac{1}{2} = C}$$

So, the solution to the IVP is

$$\boxed{ye^{y} - e^{y} = \frac{1}{2}t^{2} + e^{2} - \frac{1}{2}}$$

We leave it like this because it would be impossible to actually solve for y.

## 2.5 The Total Derivative and Exact Equations

Before we introduce the next kind of DE that is relatively straightforward to solve, we first have to jump back into some **m**ulti -**v**ariable **c**alculus and learn about a new concept. Really, it is just an extension of **c**hain **r**ule , so we will use it to develop this notion!

Suppose we let $x_1 = x_1(t) = t^2 + 1$, we let $x_2 = x_2(t) = \sin(t)$, and we let $x_3 = x_3(t) = \ln(t)$.

Then, suppose that $f(t) = e^{x_1(t)} + (x_2(t))^5 + \sqrt{x_3(t)} + \tan(t)$.

$$= e^{t^2+1} + (\sin(t))^5 + (\ln(t))^{1/2} + \tan(t)$$

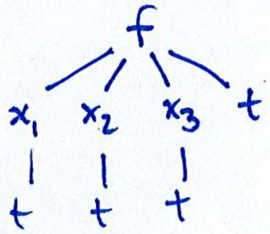
The task: Find $\dfrac{df}{dt}$. We get:

$$\frac{df}{dt} = e^{t^2+1}(2t) + 5(\sin(t))^4 \cos(t) + \frac{1}{2}(\ln(t))^{-1/2}\left(\frac{1}{t}\right) + \sec^2(t)$$

$$= e^{x_1}(2t) + 5(x_2)^4 \cos(t) + \frac{1}{2}(x_3)^{-1/2}\left(\frac{1}{t}\right) + \sec^2(t).$$

Here, we could have thought of $f$ as being a multivariable function instead, though:

Let's try it again. Start simply by writing:



$$f(x_1, x_2, x_3, t) = e^{x_1} + (x_2)^5 + \sqrt{x_3} + \tan(t)$$

where $x_1$, $x_2$, and $x_3$ all **d**epend on $t$. Then:

What is $\frac{\partial f}{\partial x_1}$? $e^{x_1}$

What is $\frac{\partial f}{\partial x_2}$? $5x_2^4$

What is $\frac{\partial f}{\partial x_3}$? $\frac{1}{2}x_3^{-1/2}$

What is $\frac{\partial f}{\partial t}$? $\sec^2(t)$

Using this, we can rewrite our result on the previous page:

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} + \frac{\partial f}{\partial t}$$

To this point, we've only known that we can use partial derivatives on multivariable functions. But in this example, we came up with an expression for $\frac{df}{dt}$. This is known as the **t**otal **d**erivative with respect to $t$.