The Method of Undetermined Coefficients can deal with multiplicative combinations of functions in the forcing term, too. To deal with these:

- If $g(x)$ has a term that is a product of an exponential and a sin or cos function, add two terms to your $y_p$ with the exponential multiplied by each trig function. For example, if $e^x \cos(x)$ appears on the RHS, add $A_1 e^x \sin(x) + A_2 e^x \cos(x)$ to $y_p$.
- If $g(x)$ has a polynomial of degree $n$ multiplied by an exponential function $h(x)$, add $n+1$ terms to $y_p$: the exponential multiplied by each subsequent lower power of $x$ starting with $n$.
- If $g(x)$ has a polynomial of degree $n$ multiplied by a sin or cos function, do the same thing as the last point, but *once each* for the appropriate sin function and cos function.

Phew! Generally, you will simply want to use these "rules" only as a guideline. Mostly, it comes down to practice and common sense rather than memorization!

**Example 2.** *Find the general solution to*

$$y'' + y = x^2 e^{2x}$$

First, find $y_h$

Characteristic Eq: $r^2 + 1 = 0$

$\rightarrow r = \pm j$ So, $\boxed{y_h = C_1\cos(x) + C_2\sin(x).}$

Now find $y_p$:

Assume: $y_p = A_1 x^2 e^{2x} + A_2 x e^{2x} + A_3 e^{2x}$

Then: $y_p' = A_1\left(2xe^{2x} + 2x^2 e^{2x}\right) + A_2\left(e^{2x} + 2xe^{2x}\right) + 2A_3 e^{2x}$

$$y_p'' = A_1\left(2e^{2x} + 4xe^{2x} + 4xe^{2x} + 4x^2e^{2x}\right) + A_2\left(2e^{2x} + 2e^{2x} + 4xe^{2x}\right) + 4A_3e^{2x}$$

Sub in:

$$A_1\left(\underset{\circ\circ\circ}{2e^{2x}} + \underline{8xe^{2x}} + \utilde{4x^2e^{2x}}\right) + A_2\left(\underset{\circ\circ\circ}{4e^{2x}} + \underline{4xe^{2x}}\right) + \underset{\circ\circ\circ}{4A_3e^{2x}} + \utilde{A_1x^2e^{2x}} + \underline{A_2xe^{2x}} + \underset{\circ\circ\circ}{A_3e^{2x}} = x^2e^{2x}$$

Group by Function Type:

$$x^2e^{2x}\left(4A_1 + A_1\right) + xe^{2x}\left(8A_1 + 4A_2 + A_2\right) + e^{2x}\left(2A_1 + 4A_2 + 4A_3 + A_3\right) = x^2e^{2x}$$

Match to get a system:

$$\begin{cases} 5A_1 = 1 \\ 8A_1 + 5A_2 = 0 \\ 2A_1 + 4A_2 + 5A_3 = 0 \end{cases} \longrightarrow \boxed{A_1 = 1/5}$$

$\downarrow$

$8/5 + 5A_2 = 0 \longrightarrow \boxed{A_2 = -8/25}$

$\downarrow$

$2/5 - 32/25 + 5A_3 = 0 \longrightarrow \boxed{22/125 = A_3}$

So, $\boxed{y_p = \frac{1}{5}x^2e^{2x} - \frac{8}{25}xe^{2x} + \frac{22}{125}e^{2x}}$

and the general solution is: $\boxed{y(x) = y_h + y_p}$

The final twist: What happens if part of your assumed form for $y_p$ **a**lready **o**ccurs in $y_h$? In this case, $y_p$ is not adding anything new to the solution that does not already exist.

If, after applying all of the rules from the past few pages, any part of your $y_p$ occurs in $y_h$, the strategy is to multiply that piece of $y_p$ by a just-high-enough power of $x$ so that the resulting expression can **n**o **l**onger **b**e **f**ound in $y_h$. Then, proceed!

**Example 3.** *Find the general solution to*

$$y'' + y = \cos(t) + t^2$$

First find $y_h$: Char. Eq. $r^2 + 1 = 0 \rightarrow r = \pm j \rightarrow y_h(t) = C_1\cos(t) + C_2\sin(t)$

To find $y_p$, we assume:

$$y_p = A_1 \underset{\uparrow}{t}\cos(t) + A_2 \underset{\uparrow}{t}\sin(t) + A_3 t^2 + A_4 t + A_5$$

These are here because cos(t) and sin(t) by themselves are already in $y_h$!!

$$y_p' = A_1\left(\cos(t) - t\sin(t)\right) + A_2\left(\sin(t) + t\cos(t)\right) + 2A_3 t + A_4$$

$$y_p'' = A_1\left(-\sin(t) - \left(\sin(t) + t\cos(t)\right)\right) + A_2\left(\cos(t) + \left(\cos(t) - t\sin(t)\right)\right) + 2A_3$$

*Here's some more space...*

Sub in:

$$A_1\left(\underline{-2\sin(t)}-\underline{t\cos(t)}\right)+A_2\left(\underline{2\cos(t)}-\underline{t\sin(t)}\right)+2A_3$$

$$+A_1\underline{t\cos(t)}+A_2\underline{t\sin(t)}+A_3\underline{t^2}+\underline{A_4t}+A_5=\cos(t)+t^2$$

Group:

$$\sin(t)\left[-2A_1\right]+\cos(t)\left[2A_2\right]+\cancel{t\cos(t)\left[-A_1+A_1\right]}+\cancel{t\sin(t)\left[-A_2+A_2\right]}$$

$$+t^2\left[A_3\right]+t\left[A_4\right]+\left(2A_3+A_5\right)=\cos(t)+t^2$$

We match to get a system:

$$-2A_1=0\longrightarrow\boxed{A_1=0}$$

$$2A_2=1\longrightarrow\boxed{A_2=\tfrac{1}{2}}$$

$$\boxed{A_3=1}$$

$$\boxed{A_4=0}$$

$$2A_3+A_5=0\rightarrow 2(1)+A_5=0\rightarrow\boxed{A_5=-2}$$

We have $y_p$!

$$\boxed{y_p=\frac{1}{2}t\sin(t)+t^2-2}$$

General Solution:

$$y(t)=y_h+y_p$$

## 6.3 The Method of Variation of Parameters

Unlike the method of undetermined coefficients, this is a general method of finding $y_p$ that can **a**lways be used in theory, no matter what the form of the DE is - this includes DEs with variable coefficients. The drawback is that this method requires us to perform **i**ntegrations that may be messy or downright **i**mpossible. Here is a rough sketch for how this method works:

Consider the general $n^{th}$-order linear DE

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_1(x)y'(x) + p_0(x)y(x) = g(x).$$

Let $\{y_1, \ldots, y_n\}$ be a **f**undamental **s**olution **s**et to the corresponding homogeneous equation:

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_1(x)y'(x) + p_0(x)y(x) = 0.$$

Then, a particular solution is given by

$$y_p(x) = v_1(x)y_1(x) + \ldots + v_n(x)y_n(x),$$

It turns out that the functions $v_1, \ldots, v_n$, unknown initially, can be determined solving the system

$$\begin{aligned}
v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} &= g(x) \\
v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)} &= 0 \\
&\vdots \\
v_1' y_1' + \ldots + v_n' y_n' &= 0 . \\
v_1' y_1 + \ldots + v_n' y_n &= 0
\end{aligned}$$

Luckily, there is an algebra trick to solve this quickly. Using **C**raner's **R**ule, we obtain that

$$v_i' = \frac{W_i(y_1, \ldots, y_n)}{W(y_1, \ldots, y_n)}$$

where $W(y_1, \ldots, y_n)$ is the **W**ronskian of $y_1, \ldots, y_n$, and $W_i(y_1, \ldots, y_n)$ is the **W**ronskian of $y_1, \ldots, y_n$ except with the $i^{th}$ column replaced with $(0, 0, \ldots, g(x))^T$.

We can then **i**ntegrate each of the $v_i'$ functions we get to obtain each of the unknown functions $v_i$, thus yielding $y_p$.

**Example 4.** *Find the general solution to*

$$y'' + y = \sec^3(x).$$

First, find $y_h$: $r^2 +$ [redacted] $= 0 \rightarrow r = \pm j \rightarrow y_h = C_1 \underbrace{\cos(x)}_{"y_1"} + C_2 \underbrace{\sin(x)}_{"y_2"}$.

Now, to find $y_p$, we set:

$y_p = v_1 y_1 + v_2 y_2 = v_1 \cos(x) + v_2 \sin(x)$, our job is to figure out what the unknown functions $v_1$ and $v_2$ are!

Let's find the Wronskians we need!

$$W(\cos(x), \sin(x)) = \det\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} = \cos^2(x) - (-\sin^2(x))$$
$$= \cos^2(x) + \sin^2(x) \boxed{= 1}$$

For $W_1$, replace the 1st column by all zeroes, with bottom entry equal to the RHS of the DE!

$$W_1 = \det\begin{pmatrix} \overset{\downarrow}{0} & \sin(x) \\ \sec^3(x) & \cos(x) \end{pmatrix} = -\sin(x)\sec^3(x) \boxed{= \frac{-\sin(x)}{\cos^3(x)}}$$

$$W_2 = \det\begin{pmatrix} \cos(x) & 0 \\ -\sin(x) & \sec^3(x) \end{pmatrix} = \cos(x)\sec^3(x) \boxed{= \sec^2(x)}$$

My "v" functions are given by:

$$v_1' = \frac{W_1}{W} = \frac{-\sin(x)}{\cos^3(x)} \rightarrow v_1 = \int \frac{-\sin(x)}{\cos^3(x)}\,dx = \int -\sin(x)(\cos(x))^{-3}\,dx$$
$$= \frac{(\cos(x))^{-2}}{-2} \boxed{= -\frac{1}{2}\sec^2(x)}$$

[redacted] ↑ we won't need the +C... we'll see why.

$$v_2' = \frac{W_2}{W} = \sec^2(x) \rightarrow v_2 = \int \sec^2(x)\,dx \boxed{= \tan(x)}$$

Thus, our $y_p$ is given by

$$y_p = \underbrace{\left(-\frac{1}{2}\sec^2(x)\right)}_{v_1}\underbrace{\cos(x)}_{y_1} + \underbrace{(\tan(x))}_{v_2}\underbrace{\sin(x)}_{y_2} = \boxed{-\frac{1}{2}\sec(x) + \frac{\sin^2(x)}{\cos(x)}}$$

and the general solution is $\boxed{y(x) = y_h + y_p}$

## 7 Vibrations: An Application of Second-Order DEs

Second-order Linear DEs are useful because they model many important physical processes involving oscillations, such as:

- Motion of a pendulum
- Mass-Spring systems
- LRC electrical circuits
- ...Much more!

Analyzing the solutions of second-order linear DEs and attaching meaning is not hard to do! We will start from the simplest situation and work our way to more complicated ones.

## 7.1 The Simple Harmonic Oscillator

The simple harmonic oscillator is an "ideal" or unrealistic system that retains all of its energy for all time. The simple harmonic oscillator models such cases as:

- A pendulum swinging with zero friction
- A spring that retains all of its energy when set into motion without dissipating
- An LC circuit that contains absolutely zero resistance

All of these situations can be modeled by the DE:

$$u''(t) + \omega^2 u(t) = 0$$

where $\omega > 0$ represents the natural frequency of the system.

✍ Here, we will only show how/why this model makes a lot of sense. The textbook gives a more detailed derivation for this DE using concepts from mechanics. See Page 192 Of Boyce and DiPrima for more!

Indeed, solving this equation can be easily done: A characteristic equation is given by

$$r^2 + \omega^2 = 0$$

$$\rightarrow \quad r^2 = -\omega^2$$

$$\rightarrow \quad r = \pm j\omega$$

with roots of $r = \underline{\pm j\omega}$. This leads to a general solution of

$$u(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t). \quad *$$

In fact, given a solution of this form, it is always possible to rewrite this solution as:

$$u(t) = R\cos(\omega t - \theta),$$

where $\omega$, $\theta$, and $R$ are constants. Let's prove this useful fact now!

Start with $u(t) = R\cos(\omega t - \theta)$.

$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

$$= R\left(\cos(\omega t)\cos(\theta) + \sin(\omega t)\sin(\theta)\right)$$

$$= \underbrace{R\cos(\theta)}\cos(\omega t) + \underbrace{R\sin(\theta)}\sin(\omega t).$$

Compare with *

If we let $C_1 = R\cos(\theta)$ and $C_2 = R\sin(\theta)$, we get back to the other form!

Using this, $C_1^2 + C_2^2 = R^2\cos^2(\theta) + R^2\sin^2(\theta)$

$$= R^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = R^2(1) = R^2.$$

$$\rightarrow \boxed{R = \sqrt{C_1^2 + C_2^2}}$$

(if $R$ is positive)

and $\frac{C_2}{C_1} = \frac{R\sin(\theta)}{R\cos(\theta)} = \tan(\theta)$, so

$$\boxed{\theta = \arctan\left(\frac{C_2}{C_1}\right)}$$

The point is that solutions are always given by a simple trig function that continues to oscillate with **c**onstant **a**mplitude for all time. This represents the motion of whatever system our DE may be modeling, which continues for all time in the absence of any resistive force.

## 7.2 Damped Harmonic Oscillators

Of course, in reality, no system is **i**deal. Mass-spring systems and pendulums will face a **d**issipative force due to factors such as friction or air resistance, and electric circuits cannot truly have zero resistance. Resistance to motion means that we must introduce a term involving $u'$ to our model (representing velocity or current). The model becomes a little more complicated:

$$u''(t) + 2\lambda u'(t) + \omega^2 u(t) = 0,$$

where $\lambda > 0$ is another constant.

✍ This formulation may be written in various ways depending on the situation, grouping the constants differently for what is most convenient. Be ready to be flexible!

Here, the characteristic equation is given by:

$$r^2 + 2\lambda r + \omega^2 = 0$$

with roots of:

$$\begin{aligned} r &= \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} \\ &= -\lambda \pm \sqrt{\lambda^2 - \omega^2} \end{aligned}$$

Now, depending on the size of $\lambda$ (think: the amount of friction, etc) in comparison to $\omega$ (the natural frequency of the system), we get a few different cases.

## Case 1: $\lambda < \omega$

In this case, we have that the stuff under the root is **n**egative.

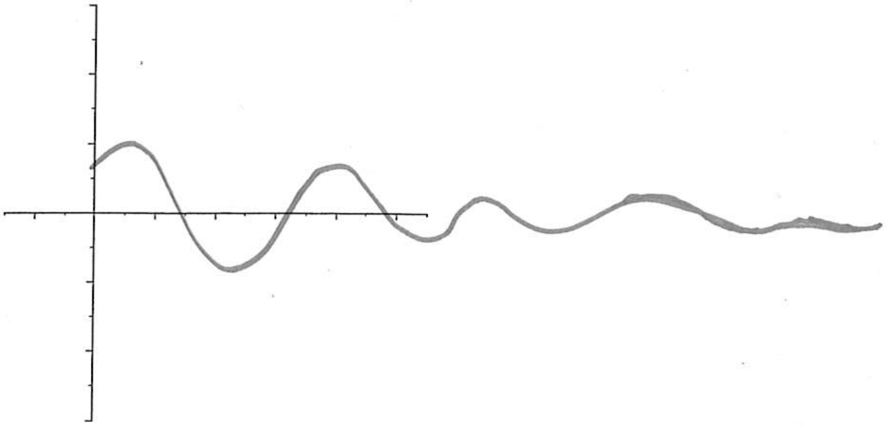
Thus, we obtain **c**omplex roots to the characteristic equation,

and hence a general solution of

roots: $-\lambda \pm \sqrt{\omega^2 - \lambda^2}\, j$

$$u(t) = e^{-\lambda t}\left(C_1 \cos\left(\left(\sqrt{\omega^2 - \lambda^2}\right) t\right) + C_2 \sin\left(\left(\sqrt{\omega^2 - \lambda^2}\right) t\right)\right).$$

We obtain oscillation for all time, but notice that the **a**mplitude includes an exponential function $e^{-\lambda t}$, where $\lambda > 0$. This means that the **a**mplitude must decrease exponentially over time, ensuring that the long-term behaviour of the solution approaches **z**ero. This is the case of **d**amped vibration, and is a very intuitive model for how dissipative forces (friction, etc) act to "slow oscillations to a stop".



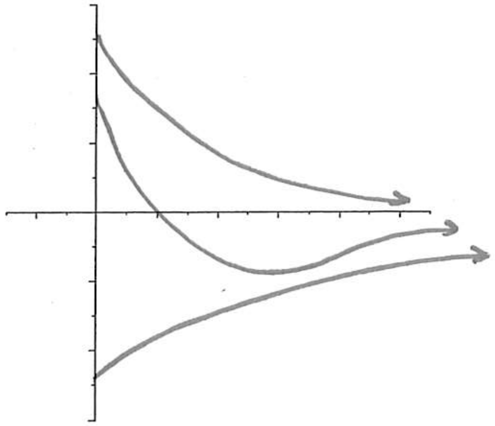
## Case 2: $\lambda > \omega$

In this case, we have that the stuff under the root is **p**ositive.

The characteristic equation has **r**eal roots, and we obtain a general solution of

$$u(t) = C_1 e^{(-\lambda+\sqrt{\lambda^2-\omega^2})t} + C_2 e^{(-\lambda-\sqrt{\lambda^2-\omega^2})t}.$$

Now, $\sqrt{\lambda^2 - \omega^2}$ is not as big in size as $\lambda$ itself. Thus, in both exponential functions, the values in front of the $t$ are negative. So, the motion similarly dies off over time, but oscillation **n**ever **o**ccurs due to the relatively heavy dissipative force; it may pass through the equilibrium position at most once, approaching equilibrium thereafter. This is the case of **o**verdamped motion.



When $\lambda$ is equal to $\omega$ exactly, it is often called **c**ritically **d**amped, representing the exact value at which the behaviour of the motion changes dramatically.

### 7.3 Forced Vibrations

When an external force is applied to a system, we can model it by including a **f**orcing **t**erm to our DE. This gives us a nonhomogeneous DE of the form:

$$u''(t) + 2\lambda u'(t) + \omega^2 u(t) = \underbrace{g(t)}_{\text{Forcing term.}},$$

As we know, the solution to any DE of this form is given by $y_h + y_p$, where $y_h$ is the solution to the corresponding homogeneous equation and $y_p$ is a particular solution. $y_h$, however, represents the solutions we just described in the previous section, in the absence of any applied force!

- $y_h$ is thus often referred to as the **f** ree **response** of the system, and is **t** ransient in that it always dies out in these models, since $\lim_{t\to\infty} y_h =$ 0 in every case. After some time has passed, $y_h$ becomes negligible in its contribution to the solution, leaving only $y_p$.

- $y_p$ is often called the **f** orced **response**. If, as $t \to \infty$, the solution continues to possess a steady oscillation or constant behaviour, that behaviour is often called the **s** teady **s** tate **s** olution of the system.

Let's see how these concepts might turn up quite naturally with a couple of examples.

**Example 1.** *The motion of a pendulum is governed by the DE*

$$x'' + \alpha x' + 4x = g(t).$$

*where $x = x(t)$ represents distance in cm away from the equilibrium (resting) position at time $t$ in seconds. $\alpha \in \mathbb{R}$ is unknown. Assume that displacement to the right is positive. The pendulum is held 1 cm to the left of equilibrium initially, and given an initial push to the right at a velocity of 2 cm/s.*

*a) Set up an IVP modeling this situation.*

$$x'' + \alpha x' + 4x = g(t).$$
$$x(0) = -1$$
$$x'(0) = 2$$

*b) Which term in the DE represents the damping term? Which term represents the forcing term?*

Damping: "$+\alpha x'$"  Forcing: "$g(t)$"

*c) For what value(s) of $\alpha$ does the motion of the pendulum exhibit oscillation that decays over time?*

Implies we need $e^{(\text{negative})t}$ (sin and cos) → we need complex roots with negative real part.

Characteristic Equation:

$$r^2 + \alpha r + 4 = 0$$

$$\rightarrow r = \frac{-\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

Complex roots when $\alpha^2 < 16 \rightarrow -4 < \alpha < 4$ and

Negative real part if $\alpha > 0$

Putting both together

$$\boxed{0 < \alpha < 4}$$