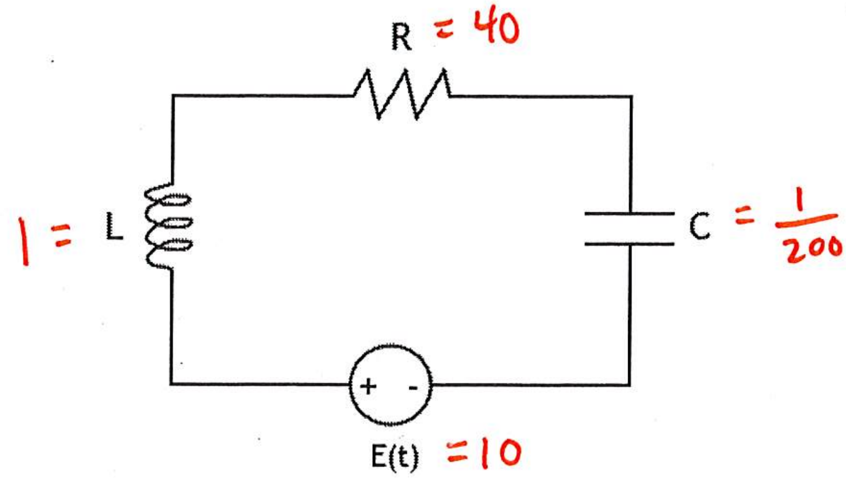
Example 2. *Consider the following circuit diagram:*



Here, the circuit has inductance $L = 1$ *H, resistance* $R = 40\ \Omega$, *capacitance* $\frac{1}{200}$ *F, and a battery that applies a voltage* $E(t)$ *of a constant* 10 *V. At time* $t = 0$, *the switch is closed (so at that very instant, assume there is no initial voltage or current).*

a) Construct an IVP that models this situation with the voltage across the capacitor $V_c(t)$ *as the dependent variable.*

$$V_L + V_R + V_c = E(t)$$

Need these in terms of V_c!

Well, $V_R = R\,i(t)$, $V_c = \frac{1}{C}\int_0^t i(\tau)\,d\tau$

$$\boxed{V_R = RC\frac{dV_c}{dt}}$$

$$\frac{dV_c}{dt} = \frac{1}{C}i(t)$$

$$\to C\frac{dV_c}{dt} = i(t)$$

$$\to C\frac{d^2V_c}{dt^2} = \frac{di}{dt}$$

and $V_L = L\frac{di}{dt}$

so $\boxed{V_L = LC\frac{d^2V_c}{dt^2}}$

Thus, $LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = E(t)$

$$\to \frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = \frac{1}{LC}E(t)$$

$$\boxed{\begin{matrix} V_c'' + 40V_c' + 200V_c = 2000 \\ V_c(0) = 0 \quad V_c'(0) = 0 \end{matrix}}$$

b) *Solve your IVP.*

Nonhomogeneous! First, find "y_h" $\rightarrow$ Char. Eq:

$$r^2 + 40r + 200 = 0$$

$$r = \frac{-40 \pm \sqrt{1600 - 800}}{2}$$

$$= -20 \pm \frac{10\sqrt{8}}{2}$$

$$\boxed{= -20 \pm 10\sqrt{2}}$$

So
$r = -20 + 10\sqrt{2}$
or $-20 - 10\sqrt{2}$.

So, $\boxed{y_h = C_1 e^{(-20+10\sqrt{2})t} + C_2 e^{(-20-10\sqrt{2})t}}$ Overdamped!

Now find "y_p". Using undetermined coeffs,

Assume $y_p = A_1 \rightarrow y_p' = 0 \longrightarrow y_p'' = 0$.

Sub in:

$$200A_1 = 2000 \rightarrow A_1 = 10$$

Thus $y_p = 10$ and $V_c(t) = y_h + y_p$

$$= C_1 e^{(-20+10\sqrt{2})t} + C_2 e^{(-20-10\sqrt{2})t} + 10.$$

Now apply the ICs:

$$V_c(0) = 0 \rightarrow C_1 + C_2 + 10 = 0$$

$$V_c'(0) = 0 \rightarrow (-20+10\sqrt{2})C_1 + (-20-10\sqrt{2})C_2 = 0$$

After solving using substitution, etc, we should get

$$\boxed{C_1 = \frac{-10-5\sqrt{2}}{\sqrt{2}}} \qquad \boxed{C_2 = \frac{10-5\sqrt{2}}{\sqrt{2}}}$$

$$\boxed{\therefore V_c(t) = \frac{-10-5\sqrt{2}}{\sqrt{2}} e^{(-20+10\sqrt{2})t} + \left(\frac{10-5\sqrt{2}}{\sqrt{2}}\right) e^{(-20-10\sqrt{2})t} + 10}$$

One last note: If the natural frequency, ω, of a system matches (or is close to) the frequency of $g(t)$, wild behaviour can occur!

For instance, in the undamped case, suppose that y_h possesses $\sin(2t)$ and $\cos(2t)$, while $g(t)$ is also $\cos(2t)$. In this case y_p has a form that would require <u>$t\cos(2t)$</u> and <u>$t\sin(2t)$</u> terms. As $t \to \infty$, the solutions must grow **l**<u>arger</u> and **l**<u>arger</u> in amplitude. This is the case of **r**<u>esonance</u>.

Eventually, **r**<u>esonant</u> frequencies will cause most realistic systems to fail, as solutions grow unbounded. Thus, **r**<u>esonance</u> is an important thing to keep in mind to avoid when constructing buildings or bridges that may twist or pull with the wind or other forces that they may be subjected to!

8 Laplace transforms

8.1 Definition

Laplace Transforms are an important integral transform used in DEs. First, some definitions!

Consider a function $f(t)$ for $t \geq 0$. Then the **Laplace transform** of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is given by

$\mathcal{L}$ $\mathcal{L}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt,$$

provided that the improper integral **c**onverges. The following criteria guarantee the **c**onvergence of this integral:

- $f(t)$ must be **p**iecewise -**c**ontinuous;
  * continuous except at a finite (or countable**) number of jump discontinuities.
- We have that $|f(t)| \leq Ke^{at}$ for $t \geq M$, where K, a, M are constants. (Essentially, this means $f(t)$ can grow **n**o **m**ore **q**uickly than an exponential).
  x^x, functions with an asymptote.

If these two hypotheses hold, then the Laplace Transform of $F(t)$ exists for $s > a$.

8.2 Laplace transforms of some basic functions

Laplace Transforms simply take one function $f(t)$ as input, and transform it into another function $F(s)$ as output. We may take the Laplace Transform of many different functions. Here are some examples:

Laplace transform of a constant

Let $f(t) = 1$ for $t \geq 0$. Then,

$$
\begin{aligned}
F(s) = \mathcal{L}\{1\} &= \int_0^\infty (1)e^{-st}dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_0^b \\
&= -\frac{1}{s}\underbrace{e^{-sb}}_{\to 0 \text{ for } s>0.} + \frac{1}{s}\underbrace{e^{-s(0)}}_{=1} \\
&\boxed{= \frac{1}{s}}
\end{aligned}
$$

✍ Note that this integral only converges if s is taken to be greater than 0. As long as we can find some interval of s for which the integral converges, we are good to go!

Example 1. *Find the Laplace transform of* 1.

$\mathcal{L}\{1\} = \frac{1}{s}$.

The Laplace transform is a **l**inear **o**perator because it is defined by an integral. Thus, **m**ultiplicative **c**onstant can easily move inside or outside the transform!

Example 2. *Find the Laplace transform of* -3.

$\mathcal{L}\{-3\} = -3\mathcal{L}\{1\} = -3\left(\frac{1}{s}\right) = \frac{-3}{s}$

In general, $\mathcal{L}\{k\} = \frac{k}{s}$

Laplace transform of an exponential function

Let $f(t) = e^{at}$, where a is a constant. Then,

$$\begin{aligned}
F(s) = \mathcal{L}\{e^{at}\} &= \int_0^\infty (e^{at})e^{-st}dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^b \\
&= \lim_{b\to\infty} \frac{1}{a-s}\Big[\underbrace{e^{(a-s)b}}_{\to 0 \text{ for } s>a} - \underbrace{e^{(a-s)(0)}}_{=1}\Big] \\
&\boxed{= \frac{1}{s-a}}
\end{aligned}$$

Example 3. *Find the Laplace transform of e^{-t}.*

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s-(-1)} \boxed{= \frac{1}{s+1}}$$

Example 4. *Find the Laplace transform of 1 again, but this time by thinking of "1" as e^{0t} instead.*

$$\mathcal{L}\{1\} = \mathcal{L}\{e^{0t}\} = \frac{1}{s-0} \boxed{= \frac{1}{s}}$$

Laplace transform of trig functions

Let $f(t) = \cos(\omega t)$, where ω is a constant. Then,

$$F(s) = \mathcal{L}\{\cos(\omega t)\} = \int_0^\infty (\cos(\omega t))e^{-st}\,dt.$$

This integral requires using **i**ntegration **b**y **p**arts twice. It *is* doable (**For You to Try**), but instead, for fun, let's take a different approach.

Example 5. *Derive the Laplace transforms for* $\cos(\omega t)$ *and* $\sin(\omega t)$ *by considering the Laplace transform of a complex exponential function and considering Euler's Formula.*

Euler: $e^{jt} = \cos(t) + j\sin(t)$.

$$e^{\omega tj} = \cos(\omega t) + j\sin(\omega t)$$

So $\mathcal{L}\{e^{\omega tj}\} = \mathcal{L}\{e^{(\omega j)t}\} = \frac{1}{s-\omega j}\frac{(s+\omega j)}{(s+\omega j)}$

$$= \frac{s+\omega j}{s^2+\omega^2} = \underbrace{\frac{s}{s^2+\omega^2}}_{1} + j\underbrace{\frac{\omega}{s^2+\omega^2}}_{2}$$

But, $\mathcal{L}\{e^{(\omega j)t}\} = \mathcal{L}\{\cos(\omega t) + j\sin(\omega t)\}$

$$= \underbrace{\mathcal{L}\{\cos(\omega t)\}}_{\text{This must be } 1} + j\underbrace{\mathcal{L}\{\sin(\omega t)\}}_{\text{This must be } 2}$$

Two new formulas:

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

✍ So, the Laplace transform of a sin or cos function always yields a function of s that includes an irreducible quadratic (one with complex roots) in the denominator! On the other hand, the Laplace transform of an exponential function always results in a first-degree function of s in the denominator.

Example 6. *Find the Laplace transform of* $\sin(5t)$.

$$\mathcal{L}\{\sin(5t)\} = \frac{5}{s^2+25}$$

Example 7. *Find the Laplace transform of* $\cos(2t)$.

$$\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$$

Example 8. *Find the Laplace transform of*

$$y(t) = 4 + 5\sin\left(\frac{1}{2}t\right) - \frac{1}{3}\cos(\pi t).$$

$$\mathcal{L}\{y(t)\} = Y(s) = \frac{4}{s} + 5\,\frac{1/2}{s^2+1/4} - \frac{1}{3}\,\frac{s}{s^2+\pi^2}$$

This next rule will be of utmost importance when we apply the concept of Laplace transforms to differential equations!

Laplace transform of derivatives

Example 9. *Given a function $f(t)$, use integration by parts to derive the Laplace transform of its derivative. That is, find* $\mathcal{L}\{f'(t)\}$. Start with the definition:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

Let $u = e^{-st}$ $\quad \frac{dv}{dt} = f'(t)$

Then $\frac{du}{dt} = -se^{-st}$ $\quad v = f(t)$

$$\mathcal{L}\{f'(t)\} = \left[e^{-st} f(t)\right|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= \lim_{b\to\infty}\left[\underbrace{e^{-sb} f(b)}_{\to 0} - \underbrace{e^{-s(0)} f(0)}_{= f(0)}\right] + s\underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\mathcal{L}\{f(t)\}} \quad \boxed{= s\mathcal{L}\{f(t)\} - f(0)}$$

So $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Similarly, it is not hard to prove that

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

we need only to integrate by parts twice instead. In general,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 10. *Find the Laplace transform of* $\dfrac{d^3y}{dt^3}$.

$$\mathcal{L}\{y'''\} = s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)$$

Laplace transform of powers

Let $f(t) = t^n$. While the Laplace transform could again be found using integration by parts, we take a different approach to make things simpler.

From the last section, we have

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

If we substitute $f(t) = t$ into this equation, we obtain:

$$\mathcal{L}\{1\} = s\mathcal{L}\{t\} - (\cancel{t \text{ evaluated at zero}} = 0)$$

$$\frac{1}{s} = s\mathcal{L}\{t\}$$

$$\rightarrow \boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$$

If we instead substitute $f(t) = t^2$ into the equation, we obtain:

$$\mathcal{L}\{2t\} = s\mathcal{L}\{t^2\}$$

$$\frac{2}{s^2} = s\mathcal{L}\{t^2\}$$

From the last line!

$$\rightarrow \boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$$

In a similar way, $\mathcal{L}\{t^3\} = \dfrac{6}{s^4}$ (**For You to Try**), and in general, we find that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0.$$

Example 11. *Find the Laplace transform of the function* $x(t) = 3t^5$.

$$\mathcal{L}\{3t^5\} = 3\frac{5!}{s^6} \boxed{= \frac{360}{s^6}}$$

Laplace transform of $e^{at}f(t)$

If $f(t)$ is a function, then we know that the Laplace transform of f is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt, = F(s).$$

which defines a function of s. So, let $\mathcal{L}\{f(t)\} = F(s)$. Then, we have that:

$$F(s-a) = \int_0^{\infty} e^{-(s-a)t} f(t)\,dt$$

But, this can be rewritten as

$$F(s-a) = \int_0^{\infty} e^{-st} \underbrace{e^{at} f(t)}\,dt$$

which is nothing more than the Laplace transform for the function $e^{at} f(t)$!

Thus, we have the property that

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

where $F(s)$ is the Laplace transform for $f(t)$.

This may sound a little complicated, but examples will clear things up:

Example 12. *Find the Laplace transform of the function* $f(t) = te^{4t}$.

$a = 4$

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, where $\mathcal{L}\{f(t)\} = F(s)$.

$\mathcal{L}\{f(t)\} = \dfrac{1}{(s-4)^2}$ The Laplace of t, but with "s" replaced by "s−4".

Example 13. *Find the Laplace transform of the function* $y(t) = \cos(2t)e^{-\frac{1}{3}t}$.

$Y(s) = \mathcal{L}\{y(t)\} = \dfrac{s+1/3}{(s+1/3)^2 + 4}$ The Laplace transform of $\cos(2t)$, but with "s" replaced by "s + 1/3".

✍ This is just scratching the surface! We can take the Laplace transforms of many different functions, and usually people refer to a table to keep track of them all. It is *highly* recommended, though, that you keep these basic formulas at the forefront of your mind so that you do not need a table!

8.3 Inverse Laplace transforms

From the definition of Laplace transform we have that $\mathcal{L}\{f(t)\} = F(s)$. Thus, the **inverse Laplace transform** is given by

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

The inverse transform is also a **l**inear **o**perator.

The idea is straightforward: If we recognize a function as being in the form of "**t**he **L**aplace **t**ransform **o**f **s**omething", going backward to get to that "**s**omething" constitutes taking the inverse transform. We'll do a few examples.

had to come from sin

had to come from cos.

Example 14. *Calculate* $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4} + \frac{5s}{s^2+1}\right\}$.

$$\boxed{= \sin(2t) + 5\cos(t)}$$

Example 15. *Calculate* $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}+\frac{5}{s+1}\right\}$.

$$= te^{-t} + 5e^{-t}$$

Note! This comes from the fact that Laplace of t is $\frac{1}{s^2}$, then the exponential "shift" rule applies.

Example 16. *Calculate* $\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\}$.

We need Partial Fractions to write this down in a more useful form!

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

Heaviside Cover-Up: $\boxed{A=3}$

Common Denoms:

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A(s^2+4)+(Bs+C)s}{s(s^2+4)}$$

Match Powers in the Numerator:

Coeff of s^2: $8 = A + B \rightarrow 8 = 3 + B \rightarrow \boxed{B=5}$

Coeff of s: $\boxed{-4 = C}$

Therefore:

$$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s}+\frac{5s-4}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\boxed{= 3 + 5\cos(2t) - 2\sin(2t)}$$