✍ Things get much more complicated for repeated eigenvalues in higher dimensions. In the 3 × 3 case, for example, it is possible to have algebraic multiplicity of 3 but be "short" either one *or* two eigenvectors. Higher dimensions bring in yet more cases, such as repeated complex eigenvalues and so on. We'll save those cases for later math courses!

Example 5. *Find the general solution to the system*

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \boldsymbol{x}.$$

Find eigenvalues:

$$\det\begin{pmatrix} 2-\lambda & 0 \\ 3 & 2-\lambda \end{pmatrix} = 0 \rightarrow (2-\lambda)(2-\lambda) = 0 \rightarrow \boxed{\lambda = 2}, \text{ twice!}$$

Now finding eigenvectors, solve:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right)$$

From R_2, $3v_1 = 0$

$\rightarrow \boxed{v_1 = 0}$

and $\boxed{v_2 = s}$, $s \in \mathbb{R}$.

So, $\boxed{V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ ... Only one eigenvector.

We need to find a generalized eigenvector, P! P satisfies:

$$\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} P = V$$

$$\rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & 0 & 1 \end{array}\right)$$

From R_2, $3p_1 = 1 \rightarrow \boxed{p_1 = 1/3}$

and $\boxed{p_2 = s}$, $s \in \mathbb{R}$.

So, $P = \begin{pmatrix} 1/3 \\ s \end{pmatrix}$ → Choosing s to be zero, we get

$$P = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

The solution is then:

$$x(t) = c_1\begin{pmatrix}0\\1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}0\\1\end{pmatrix}te^{2t} + \begin{pmatrix}1/3\\0\end{pmatrix}e^{2t}\right].$$

(Extra room, if necessary...)

Let's attach an IC to this, for fun.

$x(0) = \begin{pmatrix}1\\-1\end{pmatrix}$ We can solve for the C_1 and C_2. Applying the IC,

$$\begin{pmatrix}1\\-1\end{pmatrix} = c_1\begin{pmatrix}0\\1\end{pmatrix} + c_2\begin{pmatrix}1/3\\0\end{pmatrix}$$

$$\to 1 = \frac{1}{3}C_2 \longrightarrow \boxed{C_2 = 3}.$$

and $\boxed{-1 = C_1}$

The solution is thus:

$$x(t) = -\begin{pmatrix}0\\1\end{pmatrix}e^{2t} + 3\left[\begin{pmatrix}0\\1\end{pmatrix}te^{2t} + \begin{pmatrix}1/3\\0\end{pmatrix}e^{2t}\right]$$

or in other words:

$$\boxed{\begin{aligned} x_1(t) &= e^{2t} \\ x_2(t) &= -e^{2t} + 3te^{2t}. \end{aligned}}$$

Now suppose we'd chosen a different P vector. For instance, if we'd chosen $s = 4$, we'd have a P of $P = \begin{pmatrix}1/3\\4\end{pmatrix}$ and a solution:

$$x(t) = c_1\begin{pmatrix}0\\1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}0\\1\end{pmatrix}te^{2t} + \begin{pmatrix}1/3\\4\end{pmatrix}e^{2t}\right].$$

Applying the same IC of $x(0) = \begin{pmatrix}1\\-1\end{pmatrix}$, we get:

$$\begin{pmatrix}1\\-1\end{pmatrix} = c_1\begin{pmatrix}0\\1\end{pmatrix} + c_2\begin{pmatrix}1/3\\4\end{pmatrix}$$

or, $1 = \frac{1}{3}C_2 \to \boxed{C_2 = 3}$

$-1 = c_1 + 4C_2 \longrightarrow -1 = c_1 + 4(3)$

$\to -1 = c_1 + 12 \to \boxed{c_1 = -13}$

The solution is thus:

$$x(t) = -13\begin{pmatrix}0\\1\end{pmatrix}e^{2t} + 3\left[\begin{pmatrix}0\\1\end{pmatrix}te^{2t} + \begin{pmatrix}1/3\\4\end{pmatrix}e^{2t}\right].$$

In other words,

$$\boxed{x_1(t) = e^{2t}}$$

$$x_2(t) = -13e^{2t} + 3te^{2t} + 12e^{2t}$$

$$\boxed{x_2(t) = -e^{2t} + 3te^{2t}}$$

We get the same thing as we got before!

→ Feel free to choose any valid P!

11 Fourier Series

We will end the course by introducing an interesting topic that will likely come up a lot, especially when studying signal processing or partial differential equations. This is conceptually very similar to **T**aylor **s**eries ! But before we can begin, we are going to pay some old friends, the **s**ine and **c**osine functions, a visit.

11.1 Sine and Cosine: Periodic Functions

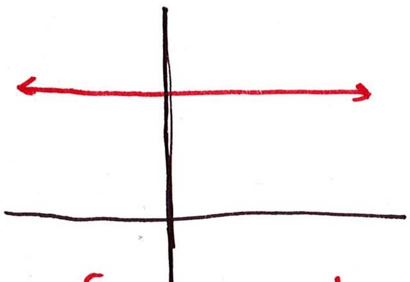
A function f is called **p**eriodic with **p**eriod T, where $T > 0$, as long as

- If x is in the domain of f, then $x + T$ is in the domain of f too; and
- $f(x + T) = f(x)$ for any x in the domain of f.

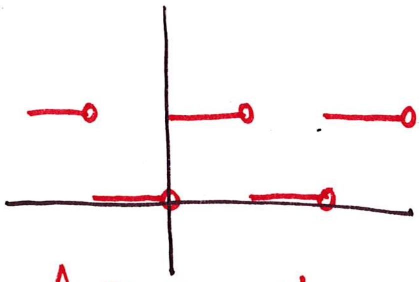
A cool fact: Suppose that f and g are both periodic functions with period T. Then it is not hard to show that:

- The **p**roduct $(fg)(x)$ is also periodic with period T; and
- Any **l**inear **c**ombination $c_1 f(x) + c_2 g(x)$ is also periodic with period T.

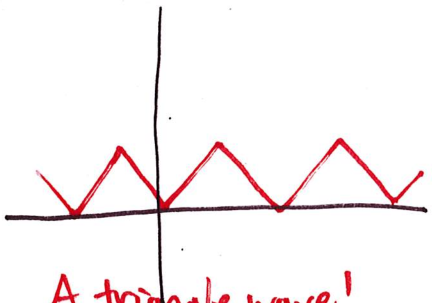
All of the basic trig functions are periodic, but they aren't the only periodic functions. Can you think of any others?



Constant Function!



A square wave!



A triangle wave!

Our sine function, $f(x) = \sin(x)$ is periodic with period 2π.

But it is also periodic with period 4π or 6π or even

$2k\pi$! Of course, we have that any periodic function with period
k is a positive integer.
T is also periodic with period kT, where k is a positive integer.

The smallest value of T for which $f(x + T) = f(x)$ is called the **fundamental period**. So for $\sin(x)$, the fundamental period is

2π. (also called "prime period")

Back to our basic trig functions: We recall that the transformed function, $f(x) = c\sin(k(x-p)) + q$ is periodic with period $\underline{2\pi/k}$, and the same thing goes for similar cosine functions. This fact will come in very handy for us in the near future!

Example 1. *Suppose that we have constants m and L such that m is a positive integer and $L > 0$ is a real number. Show that the functions $\sin\left(\frac{m\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are periodic with period $2L$, for any choice of m and L.*

$\sin\left(\frac{m\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are periodic with fundamental period

$$\frac{2\pi}{m\pi/L} = \frac{2L}{m}$$

We know, from the last page, that if a function is periodic with period $\frac{2L}{m}$, it is also periodic with period

$$\frac{2L}{m}(m) = 2L$$, if m is an integer (which it is!)

$\therefore$ proven!

11.2 Orthogonal Functions

The **d**ot **p**roduct, also called the **i**nner **p**roduct, can be defined for vectors. To find the dot product of two vectors in $\mathbb{R}^n$, we **m**ultiply **c**omponentwise and add up the result. For example,

$$(4, 2, -3) \cdot (1, 2, 3) = (4)(1) + (2)(2) + (-3)(3) = -1$$

It will now be useful for us to **g**eneralize that idea to functions. Given two real-valued functions $u(x)$ and $v(x)$, the **i**nner **p**roduct of u and v on $x \in [\alpha, \beta]$ is defined by:

$$\langle u, v \rangle = \int_{\alpha}^{\beta} u(x)v(x)dx.$$

Two functions u and v are **o**rthogonal on $x \in [\alpha, \beta]$ if $\langle u, v \rangle = 0$. A set of functions is called **m**utually **o**rthogonal if each distinct pair of functions in the set is orthogonal on $x \in [\alpha, \beta]$.

Example 2. *Show that the functions* $\cos(x)$ *and* $\sin(x)$ *are orthogonal on the interval* $x \in [-\pi, \pi]$.

$$\langle \cos(x), \sin(x) \rangle = \int_{-\pi}^{\pi} \cos(x)\sin(x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\sin(2x)\,dx$$

using a trig identity

$$= -\frac{1}{4}\Big[\cos(2x)\Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{4}\Big[\underbrace{\cos(2\pi)}_{=1} - \underbrace{\cos(-2\pi)}_{=1}\Big]$$

$$\boxed{= 0} \quad ✓$$

In fact, we can go much further: The set of functions $\cos\left(\frac{m\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ form a mutually orthogonal set of functions on the interval $x \in [-L, L]$. This is because, by integration, we can find that:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \text{ for all } m \text{ and } n.$$

We just proved on the last page that these functions are periodic with period 2L no matter what m is. Thus, the interval $x \in [-L, L]$ constitutes a period of these functions! A consequence is that if we integrate any such product of trig functions over their period, we can do so **e**xtremely **e**asily.

Example 3. *Evaluate the following integrals without doing much work at all:*

$\sin\left(\frac{m\pi x}{L}\right)$ for first

$= \sin(3x) \Rightarrow L = \pi$ and $m = 3$.

$\sin(4x)$, same idea.

a) $\displaystyle\int_{-\pi}^{\pi} \underline{\sin(3x)}\,\underline{\sin(4x)}\,dx$

These don't match!

$= 0$

b) $\displaystyle\int_{-\pi}^{\pi} \cos(\theta)\cos(5\theta)\,d\theta = 0$

c) $\displaystyle\int_{-1}^{1} \cos(\pi x)\sin(2\pi x)\,dx = 0$

d) $\displaystyle\int_{-\pi}^{\pi} \cos^2(t)\,dt = \int_{-\pi}^{\pi} \underline{\cos(t)}\,\underline{\cos(t)}\,dt$

These match!

$= \pi$

$\cos\left(\frac{m\pi t}{L}\right) = \cos(t)$ if $L = \pi$ and $m = 1$

e) $\displaystyle\int_{-1}^{1} \sin^2(\pi t)\,dt = 1$

f) $\displaystyle\int_{-2}^{2} \sin^2(\pi t)\,dt = 2$

11.3 Fourier Series

With Taylor series, our goal is to represent a function by an **i**nfinite **p**olynomial. For some interval of convergence, we are able to say that a function $f(x)$ is equal to its Taylor representation, determined by a selection of **c**oefficients in front of the powers of x.

The aim of Fourier Series is **v**ery **s**imilar! The difference is that Fourier series represent a function $f(x)$ by not an infinite series involving powers of x, but by an infinite series involving sin and cos functions. Since the functions making up a Fourier series are all periodic, Fourier representations may be found for **p**eriodic functions of all sorts.

Now, consider a function $f(x)$ with period $2L$ and set it equal to an **i**nfinite **s**eries of trig functions:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right).$$

We assume **c**onvergence of this series for now, though it turns out that this is not an issue. Our goal is to determine the "a" and "b" coefficients that make this expression true.

To do this, consider any *fixed* positive integer n, and first multiply this equation by $\cos\left(\frac{n\pi x}{L}\right)$. We get

$$\cos\left(\frac{n\pi x}{L}\right) f(x) = \cos\left(\frac{n\pi x}{L}\right)\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_m \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right)$$

Then, integrate each side from $-L$ to L:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right) f(x)\,dx = \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\frac{a_0}{2}\,dx}_{=0,\text{ we're integrating over a period of cos!}} + \sum_{m=1}^{\infty}\left[\overbrace{\int_{-L}^{L} a_m \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}^{0\text{ for every } m\neq n \text{ and } L \text{ (times am) only if } m=n} + \underbrace{\int_{-L}^{L} b_m \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=0\text{ for every } m!}\right]$$

Evaluating these integrals is not that hard at all because of the

o rthogonality properties that we just discussed!

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx = 0 + a_n L$$

The expression drastically reduces to the following, bringing us to a formula:

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx}$$

This gives us a way to figure out what our "a" coefficients should be!

We can find a formula for b_n by instead multiplying the series by $\sin\left(\dfrac{n\pi x}{L}\right)$ and integrating from $-L$ to L. The process is very similar and we get a similar-looking formula! This is **For You to Try**.

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx}$$

This leaves one last coefficient: a_0! We can find a_0 by not multiplying the series by *anything* and just integrating both sides between $-L$ and L. Since all of the basic cosine and sine functions are being integrated over entire periods, we automatically know that the functions inside the sum must give us a result of **z**ero !

Integrating, we get:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2}\,dx$$

$$= \frac{a_0}{2}x\Big|_{-L}^{L} = \frac{a_0}{2}(L-(-L)) = \frac{a_0}{2}(2L)$$

$$\rightarrow \boxed{\frac{1}{L}\int_{-L}^{L} f(x)\,dx = a_0}$$

In the end, we obtain these three very useful formulas:

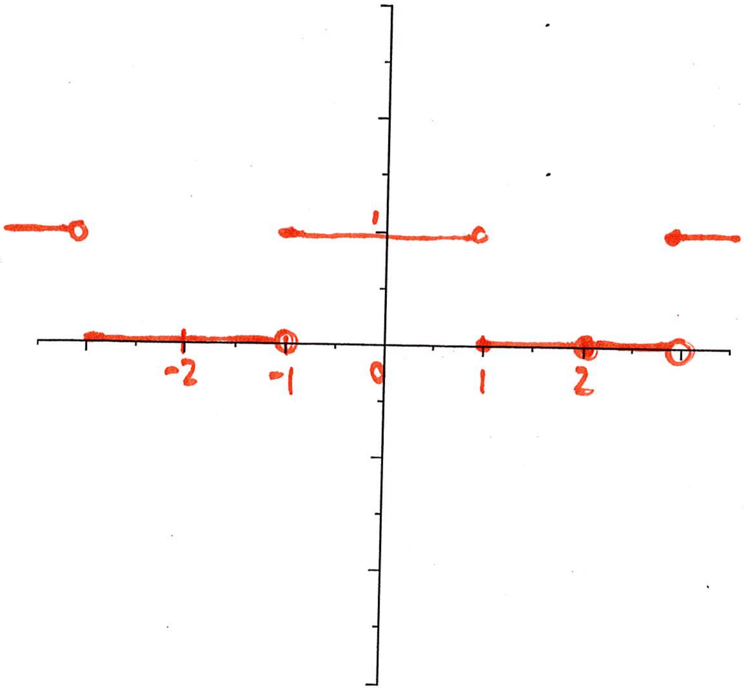
- $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$
- $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$
- $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx$

Example 4. *Find the Fourier Series for*

$$f(x) = \begin{cases} 0, & -2 \le x < -1 \\ 1, & -1 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases},$$

with periodicity given by $f(x) = f(x+4)$.

First, let's draw the graph of $f(x)$:



Now, let $L =$ 2, since the period is 4. Then, we must find the coefficients $a_n, n = 0, 1, 2, \ldots$ and $b_n, n = 1, 2, \ldots$. Let's use our new formulas and some integration muscle!

a_o:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}(\text{Area of that rectangle})$$

$$= \frac{1}{2}(2\cdot 1)\boxed{= 1.}$$

a_n:

$$a_n = \frac{1}{2}\int_{-2}^{2}\cos\left(\frac{n\pi x}{2}\right)f(x)\,dx = \frac{1}{2}\int_{-1}^{1}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\,\frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right|_{-1}^{1}$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \underbrace{\sin\left(-\frac{n\pi}{2}\right)}_{\text{Use a trig identity!}}\right]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{\begin{aligned} &= 0, \text{ if } n \text{ is even.}\\ &= \frac{2}{n\pi}, \text{ if } n = 1, 5, 9, \ldots\\ &= -\frac{2}{n\pi}, \text{ if } n = 3, 7, 11, \ldots \end{aligned}}$$

b_n:

$$b_n = \frac{1}{2}\int_{-2}^{2}\sin\left(\frac{n\pi x}{2}\right)f(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{2}\,\frac{2}{n\pi}\left[\cos\left(\frac{n\pi x}{2}\right)\right|_{-1}^{1}$$

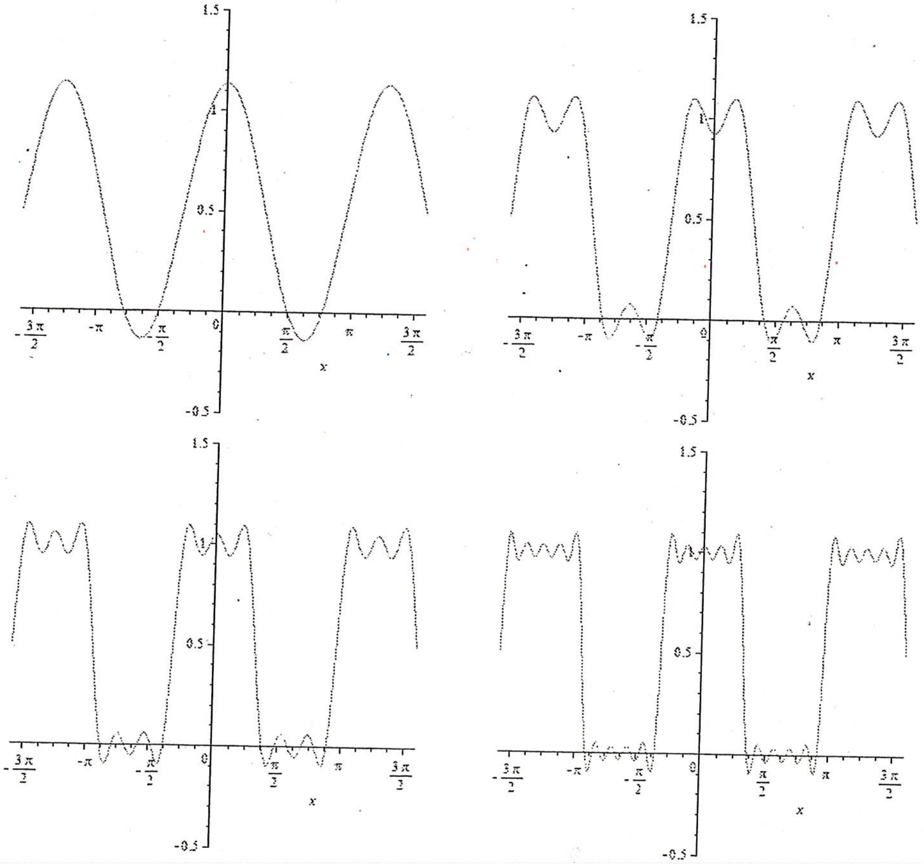
$$= -\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \underbrace{\cos\left(-\frac{n\pi}{2}\right)}_{\text{Trig identity!}}\right] = -\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right] = 0$$

$$\longrightarrow \boxed{b_n = 0 \quad \text{for any } n!}$$

Therefore, the Fourier Series for this function is:

$$f(x) = \underbrace{\frac{1}{2}}_{\frac{a_0}{2}} + \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right) + \frac{2}{5\pi}\cos\left(\frac{5\pi x}{2}\right) - \frac{2}{7\pi}\cos\left(\frac{7\pi x}{2}\right) + \dots$$

✍ In the following figures, we can easily see how taking more and more terms in the series causes the resulting functions to approach the shape of $f(x)$.



Example 5. *Find the Fourier Series for the function*

$$f(x) = 14\cos(2x) + 3\sin(5x) - 6\sin(20x).$$

If I want to write f(x) as a Fourier Series:

Choosing $L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mx) + b_m \sin(mx)\right)$$

I can choose $a_2 = 14$, $b_5 = 3$, $b_{20} = -6$.

and all other coefficients to be $= 0$ in order to get the exact function back.

✍ This is analogous to the case of trying to find the Taylor Series for a function that is polynomial in the first place. You don't need an infinite series of powers of x if your function is made of just powers of x to begin with!