4 Introduction to Higher-Order DEs

To this point, we have only discussed methods of solving **f**irst-**o**rder DEs, but higher-order DEs are very important too. Before we get into discussing how to solve them, we need to develop some **n**ew **t**heory.

4.1 Functional Notation

A **functional** is an operator or map whose **d**omain and **r**ange are in the space of **f**unctions. Think of them like "functions of functions"; i.e., they transform functions into other functions. For example, we could write $H[f] = g$ if f and g are functions with H transforming f into g.

A **linear operator** L is a functional for which the following property holds: Given functions f_1, f_2, and constants c_1, c_2, then

$$L[c_1 f_1 + c_2 f_2] = c_1 L[f_1] + c_2 L[f_2].$$

This is not a new concept for you, believe it or not! You already know of some functionals. For example, when you take the **d**erivative of a function, you are really applying a functional to it! We can see that the act of taking a **d**erivative transforms a **d**ifferentiable function into another function. The **i**ntegral operator is another.

Assume that y and x are the dependent and independent variables respectively. We introduce the following **differential operator** notation, which we will use occasionally for the coming section:

$$Dy = D[y] = \frac{dy}{dx},$$

where x is the independent variable. The notation $D^n[y]$ or $D^n y$ may be used to represent the n^{th} derivative of y with respect to x. The differential operator is a **l**inear **o**perator, which is easy to show using our derivative rules.

✍ In fact, our earlier definition of "linear DE" is in fact more solidly defined using linear operators. We just didn't have the tools yet!

Example 1. *Write the following DEs using operator notation:*

a) $y''' = \sin(t)$ $D^3 y = \sin(t)$

b) $u'' + 5u' + 6u = 0$ $D^2u + 5Du + 6u = 0$

$(D^2 + 5D + 6)u = 0$

$(D+2)(D+3)u = 0$

✍ It is true that linear operators can **d** istribute , just like variables can — and in fact, we can go a step further and "factor" expressions containing them in a familiar way. Nevertheless, be careful: they are not variables and do *not* represent any kind of quantity on their own.

c) $x^{(4)} - 2x'' + x = 0$ $D^4x - 2D^2x + x = 0$

$(D^4 - 2D^2 + 1)x = 0$

$(D^2 - 1)^2 x = 0$

$(D-1)^2(D+1)^2 x = 0.$

4.2 Second-Order Linear DEs Theory

Second order linear DEs can always be arranged into the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, and g are **c**ontinuous on at least some open interval. If we use operator notation, and let

$$L = D^2 + pD + q,$$

then we can rewrite our DE very simply as

$$L[y] = g(t).$$

L is a linear operator due to the linear form of the DE and linearity of the differential operator.

For now, we consider **h**omogeneous second-order linear DEs; that is, the case where $g(t) = 0$.

Theorem. (Superposition Principle of Solutions to Linear Homogeneous ODEs)

Suppose that $y = y_1(t)$ and $y = y_2(t)$ are two solutions to the DE

Remember: $L = D^2 + pD + q$

$(\text{So } y'' + py' + qy = 0)$

$$L[y] = 0.$$

where p and q are continuous on an open interval I, and L is defined as on the previous page. Then,

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is **a** lso a solution to the DE, where c_1 and c_2 are

a rbitrary **c** onstants .

Proof: $y_1(t)$ is a solution to $L[y] = 0$. So, $L[y_1] = 0$.

Similarly, $L[y_2] = 0$.

But then, multiplying by any constant gives:

$c_1 L[y_1] = 0$ and $c_2 L[y_2] = 0$.

Adding these together gives:

$$c_1 L[y_1] + C_2 L[y_2] = 0$$

and because of the linearity of L, we get:

$$L\left[c_1 y_1 + c_2 y_2\right] = 0$$

and this implies that $c_1 y_1 + c_2 y_2$ is also a solution to the DE!

This theorem extends to the case where we have n solutions to an n^{th} order DE! Any linear combination of those n solutions will also be a solution to the DE.

But let's play for a bit. If

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is a solution to a 2^{nd}-order DE, then suppose we had 2 initial conditions to go with this DE. These would be

$$y(t_0) = \underbrace{y_0}_{\text{Some constant}}, \qquad y'(t_0) = \overbrace{y_0'}^{\text{some other constant.}}$$

where $t_0 \in I$. We can easily apply these initial conditions to our hypothetical solution:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

This looks an awful lot like a system of equations, and so we can represent it by a **m**atrix !

Our goal in solving IVPs is always to solve for the constants, so rewriting this using **m**atrices in a format that captures this, we obtain:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

From **l**inear **a**lgebra, this system has a solution as long as the **d**eterminant of this matrix is **n**on-**z**ero!

We call this determinant the **Wronskian**, denoted $W(y_1, y_2)$. It is

here, it is evaluated at $t = t_0$

always possible to choose the constants c_1, c_2 to satsify the IVP if and only if the Wronskian $W(y_1, y_2)$ is non-zero at t_0.

Example 2. *Find the Wronskian $W(f_1, f_2)$ evaluated at $x = 0$ if $f_1(x) = \sin(2x)$ and $f_2(x) = x\sin(2x)$.*

$$W(\sin(2x), x\sin(2x)) = \det\begin{pmatrix} \sin(2x) & x\sin(2x) \\ 2\cos(2x) & \sin(2x) + 2x\cos(2x) \end{pmatrix}$$

$$= \sin(2x)\left[\sin(2x) + \cancel{2x\cos(2x)}\right] - \cancel{2x\sin(2x)\cos(2x)}$$

$$= \sin^2(2x)$$

So, evaluated at $x = 0$, we get

$$\sin^2(0) = 0$$

✍ The definition of the Wronskian can easily be extended to greater than two functions. In order to take the Wronskian of three functions, simply take the derivative of the functions that are involved **t**wice to fill in the third row.

Example 3. *Find the Wronskian* $W(f_1, f_2, f_3)$ *if* $f_1(x) = e^{-x}$, $f_2(x) = e^{3x}$ *and* $f_3(x) = x$.

$$W(e^{-x}, e^{3x}, x) = \det\begin{pmatrix} e^{-x} & e^{3x} & x \\ -e^{-x} & 3e^{3x} & 1 \\ e^{-x} & 9e^{3x} & 0 \end{pmatrix}$$

$$= e^{-x}\left[\cancel{(3e^{3x})(0)} - (1)(9e^{3x})\right]$$

$$- e^{3x}\left[\cancel{(-e^{-x})(0)} - (1)(e^{-x})\right]$$

$$+ x\left[(-e^{-x})(9e^{3x}) - (3e^{3x})(e^{-x})\right]$$

$$= -9e^{2x} + e^{2x} - 9xe^{2x} - 3xe^{2x}$$

$$\boxed{= -8e^{2x} - 12xe^{2x}}$$

For there to be a solution to the IVP, we need that the Wronskian determinant is **n** on **-z** ero for any t_0 in the open interval I. In fact, if y_1 and y_2 are solutions to the DE, then it can be shown that the Wronskian *is* either zero for every t in the interval or it is never zero there. This leads to a new theorem.

Theorem. Suppose $y'' + p(t)y' + q(t)y = g(t)$. If $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval I, if the functions y_1 and y_2 are solutions to this DE, and if the Wronskian $W(y_1, y_2)$ is nonzero for at least one value of $t_0 \in I$, then *every* solution of the DE is given by a linear combination of y_1 and y_2.

This theorem tells us a way of determining if we've found the "**w** hole " solution to a DE. It looks like for a second-order linear DE, we need **t** wo solutions for which this **W** ronskian **d** eterminant is non-zero. That's a little awkward, though, so we're going to relate this to a **m** ore **f** amiliar concept!

Linear Dependence and Independence

While this is probably review from **l**inear **a**lgebra, we'll quickly recall a simple concept:

- Two functions f_1 and f_2 are called **linearly independent** if, for constants c_1, c_2, we have that $c_1 f_1 + c_2 f_2 = 0$ **i**f **a**nd **o**nly **i**f $c_1 = c_2 = 0$.
- Two functions f_1 and f_2 are **linearly dependent** otherwise; that is, if there exists some selection of c_1 and c_2 that satisfies $c_1 f_1 + c_2 f_2 = 0$ such that at least one of c_1 and c_2 are nonzero.

As a quick example, suppose $f_1(x) = x$ and $f_2(x) = 4x$. Then f_1 and f_2 are linearly dependent, because $c_1 x + c_2(4x) = 0$ could be satisfied using $c_1 =$ -4 and $c_2 =$ 1.

On the other hand, if $f_1(x) = x$ and $f_2(x) = \cos(x)$, then f_1 and f_2 are linearly independent. This is because there is no way to choose c_1 and c_2 to satisfy the equation $c_1 x + c_2(\cos(x)) = 0$ except to make both c_1 and c_2 equal to 0 !

So, putting everything together from the last few pages, to find the general solution to a second-order linear DE we must always find *two* **l**inearly **i**ndependent solutions, each of which satisfy the DE; the general solution is then a linear combination of both of them. Our challenge is finding these solutions!

✍ If we find two linearly independent solutions to a second-order DE, the set of solutions is sometimes referred to as a **f**undamental **s**olution **s**et.

✍ We can prove everything from the last several pages in a very similar way for n^{th}-order linear DEs. It turns out that for an n^{th}-order DE, we require n linearly independent solutions! So, the general solution would include n **a**rbitrary **c**onstants.