5.3 Extending to Higher-Order Homogeneous Linear DEs with Constant Coefficients

We can quite easily extend the cases we have just covered to higher-order equations. Consider the n^{th}-order DE given by

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = 0$$

Similar to before, we may assume a solution form of $y = e^{rx}$ and substitute in; this gives us an n^{th}-degree polynomial

characteristic **e**quation :

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

We then solve to determine the values of $r = r_1, r_2, \ldots, r_n$ that satisfy the equation. The cases are parallel to those we covered in the second-order case.

Case: $r_1, r_2, \ldots, r_n$ are real and distinct

The general solution to the DE is the linear combination of all of the corresponding exponential functions; namely

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$$

Case: Some of $r_1, r_2, \ldots, r_n$ are repeated

If a real root (say, r_1) is repeated m times, then we obtain m linearly independent solutions corresponding to that root:

$$e^{r_1 x}$$

$$x e^{r_1 x}$$

$$x^2 e^{r_1 x}$$

$$\ldots$$

$$x^{m-1} e^{r_1 x}$$

The general solution is then the linear combination of these along with the solutions corresponding to any other roots.

Case: Some of $r_1, r_2, \ldots, r_n$ are complex

Each distinct complex root $r = \alpha + j\beta$ will solutions as we would expect from the second-order case:

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

However, unlike the second-order case, we can encounter a new situation...

Case: Repeated complex roots

If a complex root $\alpha + j\beta$ is repeated m times, then we obtain linearly independent solutions corresponding to that root that are "hit" by powers of x in a similar way to what we've already seen with repeated roots:

$$
\begin{aligned}
&e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)\\
&xe^{\alpha x}\left(c_3 \cos(\beta x) + c_4 \sin(\beta x)\right)\\
&x^2e^{\alpha x}\left(c_5 \cos(\beta x) + c_6 \sin(\beta x)\right)\\
&\ldots\\
&x^{m-1}e^{\alpha x}\left(c_{2m-1} \cos(\beta x) + c_{2m} \sin(\beta x)\right)
\end{aligned}
$$

Again, the general solution is then the linear combination of these along with the solutions corresponding to any other roots.

An important note: Combinations of all of these cases can arise from the same DE!

Example 9. *Find the general solution to*

$$y^{(5)}(t) + 4y'''(t) + 4y'(t) = 0.$$

Char Eq: $r^5 + 4r^3 + 4r = 0$

$$\Rightarrow r(r^4 + 4r^2 + 4) = 0$$

$$\Rightarrow r(r^2+2)^2 = 0.$$

$\Rightarrow r = 0, r = \pm\sqrt{2}j$, repeated twice.

So $y(t) = C_1 e^{0t} + e^{0t}\left(C_2\cos(\sqrt{2}t) + C_3\sin(\sqrt{2}t)\right)$

$$+ te^{0t}\left(C_4\cos(\sqrt{2}t) + C_5\sin(\sqrt{2}t)\right)$$

$$\boxed{= C_1 + C_2\cos(\sqrt{2}t) + C_3\sin(\sqrt{2}t) + C_4 t\cos(\sqrt{2}t) + C_5 t\sin(\sqrt{2}t)}$$

Example 10. *Find the general solution to*

$$(D-3)^3(D-1)u(x) = 0.$$

Char Eq:

$$(r-3)^3(r-1) = 0$$

$$\Rightarrow r=3 \text{ , three times, } \quad r=1.$$

So

$$\boxed{u(x) = C_1 e^{3t} + C_2 t e^{3t} + C_3 t^2 e^{3t} + C_4 e^{t}.}$$