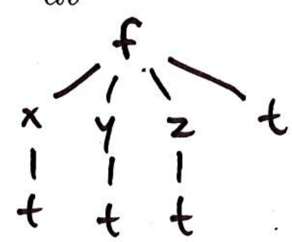
More generally now, given a multivariable function $f(x_1(t), x_2(t), \ldots, x_n(t), t)$, the total derivative $\dfrac{df}{dt}$ with respect to one of the variables (here, t) is defined by

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} + \frac{\partial f}{\partial t}.$$

Example 4. *Suppose* $x = x(t) = t^{4/5}$, $y = y(t) = \arctan(t)$, *and* $z = z(t) = t^2$.

Let $f(x, y, z, t) = t\sin(x) + x\cos(y) + xye^z + t^3$. *Find* $\dfrac{df}{dt}$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$



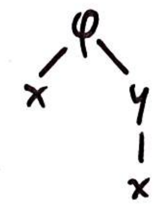
$$= \underbrace{\left(t\cos(x) + \cos(y) + ye^z\right)}_{\partial f/\partial x}\underbrace{\left(\tfrac{4}{5}t^{-1/5}\right)}_{dx/dt}$$

$$+ \underbrace{\left(-x\sin(y) + xe^z\right)}_{\partial f/\partial y}\underbrace{\left(\frac{1}{1+t^2}\right)}_{dy/dt} + \underbrace{\left(xye^z\right)}_{\partial f/\partial z}\underbrace{\left(2t\right)}_{dz/dt} + \underbrace{\left(\sin(x) + 3t^2\right)}_{\partial f/\partial t}$$

✍ You could verify that the formula works for yourself by first subbing in the expressions for x, y, and z into f and then taking $\dfrac{df}{dt}$ as usual! This is really cumbersome, though, so it's **For You to Try**.

So, question: This is all **w**ell and **g**ood, but what does this discussion have to do with differential equations?

Imagine the case where we are able to write a multivariable function as $\varphi(x, y) = C$, where C is a constant. Then, taking $\dfrac{d}{dx}$ of both sides, we obtain:



$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0,$$

So, if we were given a differential equation of that form, we know that $\varphi(x, y) = C$ will satisfy it — implicitly defining solutions to the DE. Our main problem when it comes to exact equations will always be **i**dentifying the function $\varphi(x, y)$ from looking at the differential equation.

Let's now define a new type of differential equations:

Exact **e**quations!

A DE in the form of:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called **exact** if there exists a function $\varphi(x,y)$ such that the total derivative of $\varphi(x,y)$ is

$$\frac{d\varphi}{dx} = M(x,y) + N(x,y)\frac{dy}{dx}.$$

$M(x,y)$ is then equal to $\frac{\partial \varphi}{\partial x}$, $N(x,y)$ is equal to $\partial\varphi/\partial y$, and the solution is given by $\varphi(x,y) = C$, where C is a constant.

We can test to see if an equation is exact using the following:

Test for Exactness:

The above DE is exact *if and only if*

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

✍ Note that since $M(x,y) = \dfrac{\partial \varphi}{\partial x}$ and $N(x,y) = \dfrac{\partial \varphi}{\partial y}$ this test is really just checking to see if

$$\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial^2 \varphi}{\partial x \partial y}.$$

It can be proven that this equality, related to **C**lairault's **T**heorem , is all we need to show that a DE is exact.

✍ You might also see such differential equations written in terms of the differential, like this:

$$M(x,y)dx + N(x,y)dy = 0$$

means the same as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

OK, let's finally try an example!

Example 5. *Solve the ODE given by*

$$\underbrace{3x^2y^2}_{M} + \underbrace{2x^3y}_{N}\frac{dy}{dx} = 0.$$

First, we let $3x^2y^2 = M(x,y)$ and $2x^3y = N(x,y)$, and test for exactness.

$$M_y = \frac{\partial M}{\partial y} = 6x^2y$$

$$N_x = \frac{\partial N}{\partial x} = 6x^2y \quad \checkmark$$

M_y and N_x are **e**~~xact~~qual ! So the equation is **e**xact !

This immediately tells us three crucial things:

- The solutions to this ODE are given by $\varphi(x,y) = C$.
- $M(x,y) = \dfrac{\partial \varphi}{\partial x}$;
- $N(x,y) = \dfrac{\partial \varphi}{\partial y}$;

Our task becomes finding the relation $\varphi(x,y)$.

We can find $\varphi(x, y)$ by integrating M with respect to x, and then N with respect to y:

Int. wrt x:

$$M(x, y) = 3x^2y^2 = \frac{\partial\varphi}{\partial x}, \text{ so } \varphi = x^3y^2 + C(y)$$

"+ any function of just y"

Int. wrt y:

$$N(x, y) = 2x^3y = \frac{\partial\varphi}{\partial y}, \text{ so } \varphi = x^3y^2 + C(x)$$

"+ any function of just x"

Well, φ can only be one thing. So we must make sure that the two expressions **a**gree with one another. Choose a φ that makes both expressions true:

$$\varphi = x^3y^2$$

The third point said that solutions are then given by $\varphi(x, y) = C$. Thus, for our example, the solution is:

$$x^3y^2 = C$$

Here, we could isolate for y...

$$y^2 = Cx^{-3}$$

$$y = \pm Cx^{-3/2}$$

Example 6. *Find the solution to the IVP given by:*

$$\frac{dy}{dx} = \frac{-e^x - ye^{xy}}{xe^{xy} - 2ye^{y^2} - 2y}$$
$$y(\ln(3)) = 0$$

First, rearrange into a form where we can test for exactness!

$$\underbrace{e^x + ye^{xy}}_{M} + \underbrace{\left(xe^{xy} - 2ye^{y^2} - 2y\right)}_{N}\frac{dy}{dx} = 0$$

$M_y = N_x$ is the test for exactness:

$$M_y = (1)e^{xy} + xye^{xy} \qquad N_x = (1)e^{xy} + xye^{xy}$$

They are the same! $\therefore$ the DE is exact!

I immediately know:

- $\varphi(x,y) = C$ are solutions
- $\frac{\partial \varphi}{\partial x} = M$
- $\frac{\partial \varphi}{\partial y} = N$

Apply the IC.

$$e^{(\ln 3)(0)} + e^{\ln(3)} - e^{0^2} - \cancel{0^2} = C$$

$$\cancel{1} + 3 - \cancel{1} = C \rightarrow \boxed{C=3}$$

The solution to the IVP is

$$e^{xy} + e^x - e^{y^2} - y^2 = 3$$

First, $\frac{\partial \varphi}{\partial x} = M$ means

$$\frac{\partial \varphi}{\partial x} = e^x + ye^{xy} \xrightarrow{\text{Int. wrt. } x} \varphi = e^x + e^{xy} + C(y)$$

Second, $\frac{\partial \varphi}{\partial y} = N$ means:

$$\frac{\partial \varphi}{\partial y} = xe^{xy} - 2ye^{y^2} - 2y \xrightarrow{\text{Int. wrt } y} \varphi = e^{xy} - e^{y^2} - y^2 + C(x)$$

Choose a $\varphi(x,y)$ that agrees with both, and set it equal to C.

$$\varphi = e^{xy} + e^x - e^{y^2} - y^2$$

So the general solution is: $\boxed{e^{xy} + e^x - e^{y^2} - y^2 = C}$

2.6 Using Integrating Factors to Transform "Almost Exact" DEs into Exact DEs

Let's jump right into another example.

Example 7. *Find the general solution to* $\overbrace{1+y}^{M}+\overbrace{\frac{1}{2}x}^{N}y'=0$.

We can see that here, $M(x,y)=1+y$ while $N(x,y)=\frac{1}{2}x$. So,

$M_y =$ <u>1</u> and $N_x =$ <u>1/2</u> .

This equation is not exact! There is no hope of solving it using this approach. Right? **W**<u>rong</u> !

If we were to multiply the whole DE by an **i**<u>ntegrating</u> **f**<u>actor</u> of $\mu(x)=x$, we obtain:

$$\overbrace{x+xy}^{\text{"New M"}}+\overbrace{\frac{1}{2}x^2}^{\text{"New N"}}y'=0$$

Now, $M(x,y)=x+xy$ while $N(x,y)=\frac{1}{2}x^2$. So, $M_y =$ <u>x</u>

and $N_x =$ <u>x</u> . That is, the two are now **e**<u>qual</u> !

Now the DE *is* exact, so we can proceed using the usual method (**FYTT**)! But how did we know which $\mu(x)$ to multiply by?

More generally, suppose we have a non-exact DE:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

but by multiplying by an integrating factor $\mu(x,y)$ we obtain an exact DE:

$$\overbrace{\mu(x,y)M(x,y)}^{\text{"New M"}} + \overbrace{\mu(x,y)N(x,y)}^{\text{"New N"}}\frac{dy}{dx} = 0$$

For this DE to be exact, we know from the previous section that the following must hold:

$$\frac{\partial}{\partial y}\left(\mu(x,y)M(x,y)\right) = \frac{\partial}{\partial x}\left(\mu(x,y)N(x,y)\right)$$

Our job is to find what $\mu(x,y)$ should be, but it turns out that the above equation is very difficult to solve for $\mu(x,y)$ unless μ depends **o**nly on x or **o**nly on y; not both. Let's look at both cases to try and figure out this puzzle!

Case 1: μ is a function of x only. Then, we have:

$$\frac{\partial}{\partial y}\left(\mu(x)M(x,y)\right) = \frac{\partial}{\partial x}\left(\underline{\mu(x)}\,\underline{N(x,y)}\right)$$

Let's evaluate those partial derivatives. Remember on the left-hand side that $\mu(x)$ is constant as far as y is concerned; remember

product **r**ule on the right-hand side:

$$\mu M_y = \frac{d\mu}{dx} N + \mu N_x$$

Rearrange:

$$N \frac{d\mu}{dx} = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

That means that we need for the following to be true:

$$\frac{d\mu}{dx} = \underline{\frac{M_y - N_x}{N}}\,\mu.$$

If the quantity $\dfrac{M_y - N_x}{N}$ depends only on x, then we obtain a separable and first-order linear DE that we can solve for $\mu(x)$.

Similar to the case for first-order linear equations, the solution to this DE will always be given by:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx} \tag{1}$$

Case 2: μ is a function of y only. Then, the same calculations give

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu(y).$$

If the quantity $\dfrac{N_x - M_y}{M}$ depends only on y, then we obtain a solution of

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy} \tag{2}$$

(Coming up with this formula is no harder than the other one. But yes, it's **For You to Try**!)

✍ While it isn't particularly hard to derive these formulas and solve for $\mu(x)$ from scratch, it is far, far easier to learn these two formulas and be able to use them. Check to see if:

- $\dfrac{M_y - N_x}{N}$ depends only on x. If so, then use Formula (1) to find $\mu(x)$;
- $\dfrac{N_x - M_y}{M}$ depends only on y. If so, then use Formula (2) to find $\mu(y)$;
- If *neither* of these is true, the DE can't be made exact using this approach and we should probably turn to a different method!

Finally, having found μ, **m**ultiply the DE by it. The DE should now be **e**xact, and solvable in the same manner as any **e**xact DE.

Phew! Too much talk, not enough example. Let's do an example.

Example 8. *Solve the DE given by*

$$\underbrace{y}_{M} + \underbrace{(2x - e^y)}_{N}\frac{dy}{dx} = 0.$$

Test for Exactness:

$M_y = 1$ $\quad N_x = 2$

$M_y \neq N_x$. The DE is not exact!

~~Check: Is $\frac{M_y - N_x}{N}$ just a function of x?~~

~~$= \frac{1-2}{2x-e^y} = \frac{-1}{2x-e^y}$~~

Doesn't look like it!

Is $\frac{N_x - M_y}{M}$ just a function of y?

$= \frac{2-1}{y} = \frac{1}{y}$ ✓

So an integrating factor is given by:

$$\mu(y) = e^{\int \frac{1}{y}\,dy} = e^{\ln|y| + C}.$$

After a bit of work, $\boxed{\mu(y) = y}$

So now, multiply through the DE:

$$\underbrace{y^2}_{\text{"New M"}} + \underbrace{(2xy - ye^y)}_{\text{"New N"}}\frac{dy}{dx} = 0$$

Test: $M_y = N_x$?

$M_y = 2y$ $\quad N_x = 2y$ ✓ The DE is exact!

So I immediately know:

- Solutions are $\varphi(x,y) = C$
- $\frac{\partial \varphi}{\partial x} = M$
- $\frac{\partial \varphi}{\partial y} = N$

$$\int ye^y\,dy = ye^y - e^y$$

$u = y \quad \frac{dv}{dy} = e^y$

$\frac{du}{dy} = 1 \quad v = e^y$

First $\frac{\partial \varphi}{\partial x} = y^2 \xrightarrow{\text{Int. wrt } x} \varphi = xy^2 + C(y)$

Second, $\frac{\partial \varphi}{\partial y} = 2xy - ye^y \xrightarrow{\text{Int. wrt } y} \varphi = xy^2 - ye^y + e^y + C(x)$

Find a φ that fits both expressions and set it equal to C.

$$\underbrace{xy^2 - ye^y + e^y}_{\varphi} = C$$

2.7 Solving DEs Using Substitutions

Back when we were first learning **a** ntiderivatives, we sometimes used a substitution to convert a **d** ifficult - **l** ooking integral into an integral that is **e** asier to solve.

We can do the same thing with certain differential equations! The basic flow for using this method, no matter the substitution, is to:

- Introduce a new dependent variable (often u or v is used for this purpose) that involves the original dependent variable in a way that will make the DE **s** impler ;
- Build the rest of the DE out of that new variable, making sure that the original dependent variable is **n** owhere **t** o **b** e **s** een in the new DE. To replace a derivative in this way, don't you dare forget **c** hain **r** ule !
- Find the general solution for the new DE;
- **C** onvert **b** ack to the original dependent variable in one last step.

Any kinds of substitutions could be used, so be prepared for anything. That being said, a couple of specific types come up more frequently than others.

Substituting $y(x) = v(x)x$

Example 9. *Solve*

$$\frac{dy}{dx} = \frac{x^3 + 2x^2 + y}{x}.$$

So $v = \frac{y}{x}$

First, if we let $y = v(x)x$, then $\frac{dy}{dx} =$ $\frac{dv}{dx}x + v$. Now make the substitution and solve!

$$\frac{dv}{dx}x + v = \frac{x^3 + 2x^2 + vx}{x}$$

Divide out the x!

$$\frac{dv}{dx}x + \cancel{v} = x^2 + 2x + \cancel{v}$$

$$\frac{dv}{dx} = x + 2$$

Int. both sides!

$$v = \frac{x^2}{2} + 2x + C$$

Convert back to y.

$$\frac{y}{x} = \frac{x^2}{2} + 2x + C \longrightarrow \boxed{y(x) = \frac{x^3}{2} + 2x^2 + Cx}$$

✍ This is also a first-order linear DE. Try solving it using a different approach as well, for practice!

✍ This technique is often useful for tackling DEs with **r**ational expressions on one side, with every term of the numerator and denominator containing the independent or dependent variables (or a mix), so that when we change "y" into "vx" we get things to **c**ancel, simplifying the equation. If that cancellation is not going to happen, this is *not* the technique to use, as this substitution will just make the equation **m**ore **c**omplicated!

That is, we'd use a substitution of $y = vx$ for DEs like:

$$y' = \frac{x + y}{2x + 3y}$$

...but not DEs like these:

$$y' = \frac{1 + y}{2x + 3y - 1}$$

✍ DEs like these are sometimes called "homogeneous" but this is very confusing, because this means something else to us already. Pay attention to the context, and you will be OK.

Bernoulli Equations

An interesting type of DE is a **Bernoulli Equation**. This is a DE of the form:

$$y' + p(t)y = q(t)y^n.$$

This equation can always be solved by making the substitution of $v(t) = y^{1-n}$. This substitution *always* leads us to a first-order linear DE in the new variable. To make the substitution more straightforward, it is always easiest to **d**ivide the **e**quation by y^n as a first step.

✍ These DEs are named after Jakob Bernoulli, who was also responsible for discovering the constant e. Just so you know, the "Bernoulli Equation" that you might learn about in Fluid Mechanics has nothing to do with this DE; it's due to another family member who was also a genius. There were MANY famous Bernoullis who were responsible for discovering all sorts of things in science through the 1600s and 1700s!

Example 10. *Solve the DE given by*

"$n=3$"

$$t^2y'(t) + 2ty(t) = y^3$$

Bernoulli! Divide the DE by y^3.

Let $v(t) = (y(t))^{1-3}$

or $v = y^{-2}$

$\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$

$$\boxed{-\frac{1}{2}v' = \frac{1}{y^3}y'}$$

$$t^2\frac{1}{y^3}y' + \frac{2t}{y^2} = 1$$

Substitute in and get:

$$-\frac{1}{2}t^2v' + 2tv = 1$$

It transforms into a first-order linear DE.

Standard Form:

$$v' - \frac{4}{t}v = \frac{-2}{t^2}$$

Integrating Factor will be

$$\mu = e^{\int -\frac{4}{t}\,dt} = e^{-4\ln|t| + c}$$

$$\boxed{= t^{-4}}$$

Multiply through by the integrating factor:

$$t^{-4}v' - 4t^{-5}v = -2t^{-6}$$

Product Rule backwards:

$$\left(t^{-4}v\right)' = -2t^{-6}$$

Integrate:

$$t^{-4}v = \frac{2}{5}t^{-5} + C$$

Isolate:

$$v = \frac{2}{5}t^{-1} + Ct^4$$

Convert back to y land:

$$y^{-2} = \frac{2}{5}t^{-1} + Ct^4$$

$$y^2 = \frac{1}{\frac{2}{5t} + Ct^4}$$

$$\boxed{y = \pm\sqrt{\frac{1}{\frac{2}{5t} + Ct^4}}}$$