10 An Introduction to Systems of First-Order ODEs

We have tackled differential equations in all sorts of ways. What could we possibly still have yet to explore?

Of course: Solving **m**any DEs, including **m**ore **t**han **o**ne dependent variable, at the same time!

10.1 Writing Systems of Equations in Matrix Form

Consider the system of n equations given by

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\ldots \\
a_{n1}x_1 + a_{n1}x_2 + \ldots + a_{nn}x_n &= b_n.
\end{aligned}
$$

Recall that we can always write such systems in a form that includes **v**ectors and **m**atrices instead, using the laws of linear algebra and matrix multiplication:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\underset{\sim}{b}}$$

Further, given the constants $\{a_{ij}\}$ and $\{b_i\}$, we can use **r**ow **r**eduction to solve for the vector components $\{x_i\}$ that satisfy this matrix equation, and thus the system of n equations. If we call the matrix $\mathbf{A}$, the vector on the right-hand side $\mathbf{b}$, and the solution vector $\mathbf{x}$, then we can write this equation in a compact form:

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

10.2 Eigenvalues and Eigenvectors

Eigenvalues and **eigenvectors** are extremely important in a huge number of applications, like solving systems of DEs.

Recall that if we have a matrix A, and multiply on the right by a nonzero vector V, V gets transformed into **a**nother **v**ector.

For example,

$$\begin{pmatrix} 2 & 5 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

If V, however, is an **eigenvector** of A, then it is the **v**ery **s**pecial **c**ase where it is transformed into the **s**ame **v**ector, multiplied by a constant. The constant is called an **eigenvalue** of A, with the associated eigenvector V. For example,

$$\begin{pmatrix} 2 & 5 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix};$$

That is, the vector $(0 \;\; 0 \;\; 1)^T$ is transformed into $(0 \;\; 0 \;\; 2)^T$,

which is simply 2 times the original. Thus, $(0 \;\; 0 \;\; 1)^T$ is an eigenvector of this matrix, and the eigenvalue it is associated to is 2 !

So, given a matrix, how can we find its eigenvalues and eigenvectors?

Let $\mathbf{A}$ be an $n \times n$ matrix. If we have that

$$\mathbf{Ax} = \lambda \mathbf{x},$$

then, as we just recalled, λ an **eigenvalue of A**; non-zero solution vectors are then called **eigenvectors**. Manipulating this definition, we obtain

$$\mathbf{Ax} - \lambda \mathbf{x} = \underset{\sim}{0} .$$

or factoring: $(A - I\lambda)\underset{\sim}{x} = \underset{\sim}{0}$

From linear algebra, this equation has **n** on -**z** ero solutions if and only if

$$det(\mathbf{A} \text{ - } \mathbf{I}\lambda) = 0 \; .$$

Why is this so? If the determinant is equal to **z** ero , then at least one of the rows of the matrix can be made from a **l** inear **c** ombination of the other rows. Thus, row reduction of the augmented matrix:

$$\left(\mathbf{A} - \mathbf{I}\lambda \,\middle|\, 0\right)$$

always results in a **r** ow **o** f **z** eroes , allowing for non-zero solution vectors.

So, in summary, the values of λ that make that determinant of the matrix $A - I\lambda$ equal to zero are the **e** igenvalues of A; we can then find those non-zero solution vectors, which are the **e** igenvectors of A!

Example 1. *Find the eigenvalues and eigenvectors for the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

I can find eigenvalues by solving $\det(A - I\lambda) = 0$.

$$\det\begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{pmatrix} = 0.$$

$$(1-\lambda)(4-\lambda)(6-\lambda) = 0$$

$\to \lambda = 1,\ \lambda = 4,\ \lambda = 6.$ These are my eigenvalues!

Let's find eigenvectors.

Associated to $\lambda = 1$, we must solve:

$$(A - I\lambda)V = 0$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) R_2 - R_3$$

From R_3, $5V_3 = 0 \to \boxed{V_3 = 0}$

From R_2, $3V_2 = 0 \to \boxed{V_2 = 0}$

V_1 can be anything! $\boxed{\text{Let } v_1 = s,}$ where $s \in \mathbb{R}$.

We get the eigenvector

$$V = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

If we choose s to be 1.

$$\boxed{V = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Associated to $\lambda = 4$: Solve,

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \tfrac{1}{5}R_2 \\ 2R_2 - 5R_3 \end{array}$$

From R_2, $\boxed{V_3 = 0}$

Let $\boxed{V_1 = s}$ $s \in \mathbb{R}$.

Then from R_1,

$$-3s + 2v_2 + 3(0) = 0$$

$$\boxed{V_2 = \frac{3}{2}s}$$

So $V = \begin{pmatrix} s \\ \frac{3}{2}s \\ 0 \end{pmatrix}$

ie $V = \begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

If $s = 1$ — If $s = 2$

(we only care about direction!)

$$\left.\begin{matrix}3\\5\\6\end{matrix}\right)$$

Eigenvector for $\lambda = 6$.

Solve:

$$\left(\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $\boxed{v_2 = s}$, $s \in \mathbb{R}$.

From R_2, $-2s + 5v_3 = 0$

$$\boxed{v_3 = \frac{2}{5}s}$$

From R_1, $-5v_1 + 2s + 3\left(\frac{2}{5}s\right) = 0.$

$$-5v_1 + \frac{16}{5}s = 0$$

$$\boxed{v_1 = \frac{16}{25}s}$$

$$V = \begin{pmatrix} 16/25 \\ 1 \\ 2/5 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 16 \\ 25 \\ 10 \end{pmatrix}.$$

OR. Let $\boxed{v_3 = s}$, $s \in \mathbb{R}$.

From R_2, $-2v_2 + 5s = 0$.

$$\rightarrow \boxed{v_2 = \frac{5}{2}s}$$

From R_1, $-5v_1 + 2\left(\frac{5}{2}s\right) + 3s = 0.$

$$-5v_1 + 8s = 0$$

$$\boxed{v_1 = \frac{8}{5}s}$$

$$V = \begin{pmatrix} 8/5 \\ 5/2 \\ 1 \end{pmatrix} \xrightarrow{\times 10} \begin{pmatrix} 16 \\ 25 \\ 10 \end{pmatrix}$$

10.3 Solving Linear Systems of DEs

Consider n DEs of the form:

$$\begin{aligned}
x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + g_1(t) \\
x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n(t) + g_2(t) \\
&\ldots \\
x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + g_n(t)
\end{aligned}$$

We may write this in matrix form, as

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t),$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots x_n(t))^T$, $\mathbf{g}(t) = (g_1(t), g_2(t), \cdots g_n(t))^T$ and

$$A(t) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

For this course, we will be concerned only with the case where all the entries of A are **c**onstant____ , and where the system is **h**________________ , so that $\mathbf{g} = 0$. Hence, our focus will be on systems with the form $\mathbf{x}'(t) = A\mathbf{x}(t)$. Or, writing it out:

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

"$x' = Ax$"

Much like we did for the case of solving a single DE, we search for a solution in the form of an **e**xponential____ :

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} e^{\lambda t},$$

where all of the V_i are constants. We could also write:

$$\mathbf{x} = Ve^{\lambda t}.$$

Assuming that this is a solution, let's substitute it into our system of DEs! To do this, we need the derivative of the vector $\mathbf{x} = (x_1(t), x_2(t), \ldots, x_n(t))^T$ with respect to the independent variable (here, t). We can do this by taking the derivative of each component. We obtain:

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \lambda V_1 \\ \lambda V_2 \\ \vdots \\ \lambda V_n \end{pmatrix} e^{\lambda t}$$

or

$$\mathbf{x}' = \lambda V e^{\lambda t}.$$

Substituting $\mathbf{x}$ and $\mathbf{x}'$ into our system $\mathbf{x}' = A\mathbf{x}$, we obtain:

$$\lambda V e^{\lambda t} = A V e^{\lambda t}$$

This is very reminiscent of our previous discussion about eigenvalues and eigenvectors! Since the exponential is nonzero, we can divide by it to obtain

$$\lambda V = AV$$

Rearranging,

$$AV - \lambda V = \underline{\underset{\sim}{0}}$$

or

$$(A - I\lambda)V = \underset{\sim}{0}$$

This should look familiar! Now, this equation only has a nonzero solution if:

$$det(A - I\cdot\lambda) = \underline{0}\,.$$

This equation is always an n^{th}-order polynomial, called the c<u>haracteristic</u> e<u>quation</u> (sound familiar?) that can be solved for λ to give n (possibly repeated) values λ_i.

The λ_i values that satisfy this equation are indeed the **e** igenvalues of A, and their nature determines the form of the solution. The associated **e** igenvectors V_i then appear in the various solution forms.

There are three cases for λ, and they are generally analogous to the cases for single DEs. We won't focus much on the derivation of the solution forms in this course, but there is much, much more detail to come in Math*3100 (Differential Equations II)!

Case 1: λ_i are real and distinct

For each i, we obtain a solution of the form

not an exponent. This is the "eigenvector associated with λ_i"

$$V^i e^{\lambda_i t},$$

where V^i is the eigenvector corresponding to λ_i. The general solution is then made up of a linear combination of each of these pieces with an **a** rbitrary **c** onstant in front of each.

$x_1'(t) = 3x_1(t) + x_2(t)$
$x_2'(t) = 12x_1(t) - x_2(t).$

Example 2. *Find the general solution to the system*

$$x' = \begin{pmatrix} 3 & 1 \\ 12 & -1 \end{pmatrix} x.$$

Find eigenvalues!

$$\det(A - I\lambda) = \det\begin{pmatrix} 3-\lambda & 1 \\ 12 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) - 12 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 3 - 12 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 15 = 0$$
$$\Rightarrow (\lambda - 5)(\lambda + 3) = 0$$
$$\Rightarrow \boxed{\lambda = 5}, \quad \boxed{\lambda = -3}.$$

Find eigenvectors!

For $\lambda = 5$, solve:

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ 12 & -6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $\boxed{v_1 = s}$, $s \in \mathbb{R}$.

Then $-2s + v_2 = 0$

$\rightarrow \boxed{v_2 = 2s}$

$$V^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda = -3$, solve:

$$\left(\begin{array}{cc|c} 6 & 1 & 0 \\ 12 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $\boxed{v_2 = s}$, $s \in \mathbb{R}$.

Then $6v_1 + s = 0$

$\boxed{v_1 = -\frac{1}{6}s}$

$$V^2 = \begin{pmatrix} -1/6 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

My general solution is

$$\boxed{x(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{5t} + C_2\begin{pmatrix} -1 \\ 6 \end{pmatrix}e^{-3t}}$$

Case 2: Two λ_i come as a complex pair $\alpha \pm j\beta$

In this case, the pair of complex eigenvalues will have two corresponding eigenvectors V^1 and V^2 that are **a** lso complex conjugates of one another; that is, the eigenvectors will be in the form

$$V^{1,2} = V^r \pm V^c,$$

where V^r is a vector made up of the **r** eal parts of V^1 and V^2, and V^c is a vector made up of the **i** maginary parts. We obtain two solutions, of the form

$$e^{\alpha t}(V^r \cos(\beta t) - V^c \sin(\beta t)),$$

and

$$e^{\alpha t}(V^r \sin(\beta t) + V^c \cos(\beta t)).$$

The reasoning behind this relates to **E** uler's formula, perhaps not-so-surprisingly!

Solve: $x' = \begin{pmatrix} -1 & 2 \\ -2 & -2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Find eigenvalues:

$$\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -2-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda) + 4 = 0$$

$$\rightarrow \lambda^2 + 3\lambda + 2 + 4 = 0$$

$$\rightarrow \lambda^2 + 3\lambda + 6 = 0.$$

$$\rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 4(1)(6)}}{2}$$

$$= \underbrace{-\frac{3}{2}}_{\alpha = -\frac{3}{2}} \pm \underbrace{\frac{\sqrt{15}}{2}}_{\beta = \frac{\sqrt{15}}{2}} j$$

Complex!

Find eigenvectors:

Considering $\lambda = -\frac{3}{2} + \frac{\sqrt{15}}{2} j$,

solve:

$$\left(\begin{array}{cc|c} -1 - \left(-\frac{3}{2} + \frac{\sqrt{15}}{2} j\right) & 2 & 0 \\ -2 & -2 - \left(-\frac{3}{2} + \frac{\sqrt{15}}{2} j\right) & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{15}}{2} j & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $v_1 = s$, $s \in \mathbb{R}$.

From R_1, $\left(\frac{1}{2} - \frac{\sqrt{15}}{2} j\right) s + 2 v_2 = 0$

$$\rightarrow v_2 = \frac{\left(-\frac{1}{2} + \frac{\sqrt{15}}{2} j\right) s}{2}$$

$$= \left(-\frac{1}{4} + \frac{\sqrt{15}}{4} j\right) s$$

My eigenvector is

$$V = \begin{pmatrix} 1 \\ -\frac{1}{4} + \frac{\sqrt{15}}{4} j \end{pmatrix}$$

or

$$\begin{pmatrix} 4 \\ -1 + \sqrt{15} j \end{pmatrix}$$

That means

(Extra room, if necessary...)

$$V = \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}_{V_r} + \underbrace{\begin{pmatrix} 0 \\ \sqrt{15} \end{pmatrix}}_{V_c} j$$

So, the general solution is:

$$x(t) = C_1 e^{-\frac{3}{2}t}\left[\begin{pmatrix} 4 \\ -1 \end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right) - \begin{pmatrix} 0 \\ \sqrt{15} \end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right)\right]$$

$$+ C_2 e^{-\frac{3}{2}t}\left[\begin{pmatrix} 4 \\ -1 \end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right) + \begin{pmatrix} 0 \\ \sqrt{15} \end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right)\right].$$

This was an IVP! We had initial condition $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 4 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ \sqrt{15} \end{pmatrix}$$

→ meaning:

$$0 = 4C_1 + 0C_2$$
$$1 = -C_1 + \sqrt{15}\,C_2.$$

→ $\boxed{C_1 = 0}$

→ From the second equation,

$$1 = \sqrt{15}\,C_2 \rightarrow \boxed{C_2 = \frac{1}{\sqrt{15}}}$$

So the solution to the IVP is:

$$x(t) = \frac{1}{\sqrt{15}} e^{-\frac{3}{2}t}\left[\begin{pmatrix} 4 \\ -1 \end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right) + \begin{pmatrix} 0 \\ \sqrt{15} \end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right)\right].$$

Case 3: λ_i is repeated

Suppose that λ_i has algebraic multiplicity n, meaning that it is r<u>epeated</u> <u>n</u> t<u>imes</u>. There are a few possibilities for this case:

a) n linearly independent eigenvectors $V^1 \dots V^n$ associated to the eigenvalue are found when solving

$$\left(\mathbf{A} - \mathbf{I}\lambda_i \,\middle|\, 0\right).$$

In this case, we proceed as before and obtain n solutions:

$$V^1 e^{\lambda_i t}$$

$$V^2 e^{\lambda_i t}$$

$$\dots$$

$$V^n e^{\lambda_i t}$$

Example 4. *Find the general solution to the system*

$$x' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} x.$$

Means $x_1' = 3x_1$
$x_2' = 3x_2$
"decoupled system".

Eigenvalues:

$$\det\begin{pmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} = 0$$

$\rightarrow (3-\lambda)(3-\lambda) = 0$

$\rightarrow \lambda = 3$ (twice!)

Eigenvectors associated to $\lambda = 3$:
Solve

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

I have two free variables!

Let $v_1 = s$ and $v_2 = r$

$$V = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{V_1} s + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{V_2} r$$

I get two eigenvectors!

General Solution:

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$$

b) But it can also happen that **f**ewer than n eigenvectors are found. This is problematic. Assume for the moment that our matrix is just 2×2. In that case, we would have had algebraic multiplicity of two but only found one regular eigenvector, V. This gives us one solution as usual:

$$\mathbf{x}_1 = Ve^{\lambda_i t},$$

Everything has been parallel to the single-DE situation so far. So, let's search for a second solution of the form

Another vector of constants of some kind!

$$\mathbf{x}_2 = Vte^{\lambda_i t} + \widehat{P}e^{\lambda_i t}.$$

If we substitute this into the original equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain:

First, take the derivative: $x_2' = V\left[e^{\lambda_i t} + \lambda_i t e^{\lambda_i t}\right] + \lambda_i P e^{\lambda_i t}$

Sub into $x' = Ax$: $Ve^{\lambda_i t} + V\lambda_i t e^{\lambda_i t} + \lambda_i P e^{\lambda_i t} = A\left[Vte^{\lambda_i t} + Pe^{\lambda_i t}\right].$

Rearranging and grouping by function type, we obtain

$$te^{\lambda_i t}\left[V\lambda_i - AV\right] = e^{\lambda_i t}\left[AP - \lambda_i P - V\right]$$

Remember: V was an eigenvector. By definition,

$AV = \lambda_i V \rightarrow \lambda_i V - AV = 0$.

So the LHS is equal to zero!

Continuing,

$$0 = e^{\lambda_i t}\left[AP - \lambda_i P - V\right]$$

$$\rightarrow \quad AP - \lambda_i P - V = 0$$

$$\rightarrow \quad (A - \lambda_i I)P = V$$

We finally come to:

$$(A - \lambda_i I)P = V.$$

We can solve for P by Gaussian elimination! We call P a

g<u>eneralized</u> e<u>igenvector</u>. If we need one

g<u>eneralized</u> e<u>igenvector</u> (i.e. if we have repeated

roots of multiplicity two but find only one regular eigenvector V),

we obtain one solution of the form:

$$Ve^{\lambda_i t},$$

with a second solution of

$$Vte^{\lambda_i t} + Pe^{\lambda_i t}.$$

Case 2: Two λ_i come as a complex pair $\alpha \pm j\beta$

In this case, the pair of complex eigenvalues will have two corresponding eigenvectors V^1 and V^2 that are **a**lso complex conjugates of one another; that is, the eigenvectors will be in the form

$$V^{1,2} = V^r \pm V^c,$$

where V^r is a vector made up of the **r**eal parts of V^1 and V^2, and V^c is a vector made up of the **i**maginary parts. We obtain two solutions, of the form

$$e^{\alpha t}(V^r \cos(\beta t) - V^c \sin(\beta t)),$$

and

$$e^{\alpha t}(V^r \sin(\beta t) + V^c \cos(\beta t)).$$

The reasoning behind this relates to **E**uler's formula, perhaps not-so-surprisingly!

Solve:

$$x' = \begin{pmatrix} -1 & 2 \\ -2 & -2 \end{pmatrix} x \quad , \; x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find eigenvalues:

$$\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -2-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda) + 4 = 0.$$

$$\rightarrow \lambda^2 + 3\lambda + 2 + 4 = 0$$

$$\rightarrow \lambda^2 + 3\lambda + 6 = 0.$$

$$\lambda = \frac{-3 \pm \sqrt{9 - 4(1)(6)}}{2(1)}$$

$$= \underbrace{-\frac{3}{2}}_{\alpha = -\frac{3}{2}} \pm \underbrace{\frac{\sqrt{15}}{2}}_{\beta = \frac{\sqrt{15}}{2}} j$$

Find eigenvectors:

Considering $\lambda = -\frac{3}{2} + \frac{\sqrt{15}}{2} j$,

solve:

$$\left(\begin{array}{cc|c} -1-\left(-\frac{3}{2}+\frac{\sqrt{15}}{2}j\right) & 2 & 0 \\ -2 & -2-\left(-\frac{3}{2}+\frac{\sqrt{15}}{2}j\right) & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{15}}{2}j & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $v_1 = s$, $s \in \mathbb{R}$.

From R_1, $\left(\frac{1}{2} - \frac{\sqrt{15}}{2}j\right)s + 2v_2 = 0$

$$v_2 = \left(\frac{-\frac{1}{2} + \frac{\sqrt{15}}{2}j}{2}\right)s = \left(-\frac{1}{4} + \frac{\sqrt{15}}{4}j\right)s$$

$$V = \begin{pmatrix} 1 \\ -\frac{1}{4} + \frac{\sqrt{15}}{4}j \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ -1 + \sqrt{15}j \end{pmatrix}$$

That means:

$$V = \underbrace{\begin{pmatrix}4\\-1\end{pmatrix}}_{V_r} + \underbrace{\begin{pmatrix}0\\\sqrt{15}\end{pmatrix}}_{V_c} j$$

(Extra room, if necessary...)

Recall:
Our eigenvalues were $-\frac{3}{2} \pm \frac{\sqrt{15}}{2} j$

So, the general solution is:

$$x(t) = C_1 e^{-\frac{3}{2}t}\left[\begin{pmatrix}4\\-1\end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right) - \begin{pmatrix}0\\\sqrt{15}\end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right)\right]$$

$$+ C_2 e^{-\frac{3}{2}t}\left[\begin{pmatrix}4\\-1\end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right) + \begin{pmatrix}0\\\sqrt{15}\end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right)\right]$$

This was an IVP! We had initial condition $x(0) = \begin{pmatrix}0\\1\end{pmatrix}$.

$$\begin{pmatrix}0\\1\end{pmatrix} = C_1\begin{pmatrix}4\\-1\end{pmatrix} + C_2\begin{pmatrix}0\\\sqrt{15}\end{pmatrix} \rightarrow$$

meaning:

$$0 = 4C_1 + 0C_2$$

$$1 = -C_1 + \sqrt{15}C_2.$$

$$\rightarrow \boxed{C_1 = 0.}$$

From the second equation,

$$1 = \sqrt{15}\, C_2$$

$$\rightarrow \boxed{C_2 = \frac{1}{\sqrt{15}}}$$

So the solution to the IVP is:

$$x(t) = \frac{1}{\sqrt{15}} e^{-\frac{3}{2}t}\left[\begin{pmatrix}4\\-1\end{pmatrix}\sin\left(\frac{\sqrt{15}}{2}t\right) + \begin{pmatrix}0\\\sqrt{15}\end{pmatrix}\cos\left(\frac{\sqrt{15}}{2}t\right)\right]$$

Case 3: λ_i is repeated

Suppose that λ_i has algebraic multiplicity n, meaning that it is r*epeated* *n* t*imes*. There are a few possibilities for this case:

a) n linearly independent eigenvectors $V^1 \dots V^n$ associated to the eigenvalue are found when solving

$$\left(\mathbf{A} - \mathbf{I}\lambda_i \,\middle|\, 0\right).$$

In this case, we proceed as before and obtain n solutions:

$$V^1 e^{\lambda_i t}$$

$$V^2 e^{\lambda_i t}$$

$$\dots$$

$$V^n e^{\lambda_i t}$$

Example 4. *Find the general solution to the system*

$$x' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} x.$$

means,
$x_1' = 3x_1$
$x_2' = 3x_2$
"decoupled system."

Eigenvalues:

$$\det \begin{pmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} = 0$$

$\rightarrow (3-\lambda)^2 = 0$

$\rightarrow \lambda = 3$ (twice!)

Eigenvectors associated to $\lambda = 3$:

Solve

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

I have <u>two</u> free variables!

Let $v_1 = s$, $v_2 = r$

$$V = \begin{pmatrix} s \\ r \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{V^1} s + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{V^2} r$$

I get <u>two</u> eigenvectors!

General Solution:

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$$

b) But it can also happen that f ewer than n eigenvectors are found. This is problematic. Assume for the moment that our matrix is just 2×2. In that case, we would have had algebraic multiplicity of two but only found one regular eigenvector, V. This gives us one solution as usual:

$$\mathbf{x}_1 = Ve^{\lambda_i t},$$

Everything has been parallel to the single-DE situation so far. So, let's search for a second solution of the form

Another vector of constants... ...of some kind!

$$\mathbf{x}_2 = Vte^{\lambda_i t} + Pe^{\lambda_i t}.$$

If we substitute this into the original equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain:

I need a derivative. So, first, take the derivative:

$$x_2' = V\left[e^{\lambda_i t} + \lambda_i t e^{\lambda_i t}\right] + \lambda_i P e^{\lambda_i t}$$

Sub into my DE, $x' = Ax$:

$$Ve^{\lambda_i t} + V\lambda_i t e^{\lambda_i t} + \lambda_i P e^{\lambda_i t} = A\left[Vte^{\lambda_i t} + Pe^{\lambda_i t}\right]$$

Rearranging and grouping by function type, we obtain

$$te^{\lambda_i t}\left[V\lambda_i - AV\right] = e^{\lambda_i t}\left[AP - \lambda_i P - V\right]$$

Remember, V was an eigenvector. By definition,

$$AV = \lambda_i V \rightarrow \lambda_i V - AV = 0.$$

Continuing,

We get
$$0 = e^{\lambda_i t}\left[AP - \lambda_i P - V\right]$$

$$\rightarrow 0 = AP - \lambda_i P - V$$

$$AP - \lambda_i P = V$$

$$(A - \lambda_i I)P = V.$$

We finally come to:

$$(A - \lambda_i I)P = V.$$

We can solve for P by Gaussian elimination! We call P a

g<u>eneralized</u> e<u>igenvector</u>. If we need one

g<u>eneralized</u> e<u>igenvector</u> (i.e. if we have repeated

roots of multiplicity two but find only one regular eigenvector V),

we obtain one solution of the form:

$$Ve^{\lambda_i t},$$

with a second solution of

$$Vte^{\lambda_i t} + Pe^{\lambda_i t}.$$