1 What is a Differential Equation?

1.1 Introduction

A **differential equation** or **DE** is an equation that contains **d**erivatives. Typically, a DE provides a relationship between a function and its derivatives with respect to one or more variables.

For example, consider the following equation:

$$y''(x) + y(x) = e^x$$

Reading in plain English, this equation says that a function's **s**econd **d**erivative, when added to the function **i**tself, yields e^x. What could that function be? Our job will often be to determine which function (or functions!) satisfies the DE. In doing this, we are said to **s**olve the DE, and there are many different techniques we can use to find solutions to DEs.

Differential equations are very useful for modelling real-world problems over a wide range of disciplines. For example:

- Population dynamics in **e** cological **s** ystems
- Quantities of reagents during a **c** hemical **r** eaction
- **P** ositions of **m** asses interacting in a system over time
- Charges or currents within an **e** lectrical **c** ircuit.

...and there are many, many more. Coming up with a model can help us to predict what might happen when something is built a certain way, or why something may behave in a particular manner, without having to actually use time or money to perform an experiment or build something with material cost.

Plus, as we will come to see, Differential Equations are just **s** uper **c** ool to study.

We will spend most of the course learning about ways to solve DEs. For now, though, we will start by learning some language that we will use to describe these mathematical objects.

1.2 Dependent and Independent Variables

First, all differential equations have **dependent** and **independent** variables. Let's revisit our first example:

$$y''(x) + y(x) = e^x$$

Here, we call <u>y</u> the dependent variable because the quantity <u>y</u> **l**<u>iterally</u> **d**<u>epends</u> on <u>x</u>. That means that <u>x</u> is the independent variable!

✍ To quickly determine which variable is dependent, find the variable *being differentiated*; those variables that are being differentiated *with respect to* are the independent variables.

Example 1. *Determine which variable(s) are dependent and which are independent in the following DEs:*

a) $\frac{d}{dx}\left(u\frac{du}{dx}\right) = \arctan(u)$ Dependent: u Independent: x

b) $y''' - y'' - y' - 4y = t$ Dependent: y Independent: t

c) $u_{xx} = u_t$ Dependent: u Independent: x and t.

(u_{xx} means $\frac{\partial^2 u}{\partial x^2}$, u_t means $\frac{\partial u}{\partial t}$)

Equations a) and b) are examples of **o**rdinary differential equations (**ODEs**) because they contain only ordinary derivatives as opposed to partial derivatives. ODEs always contain a **s**ingle dependent variable and a **s**ingle independent variable.

Equation c) is an example of a **p**artial differential equation (**PDE**) because it contains partial derivatives. PDEs contain **o**ne dependent variable and **m**any independent variables.

✍ PDEs are much more complicated than ODEs and far more difficult (often impossible!) to solve. For most of this course, we will concern ourselves only with ODEs.

1.3 Order of a DE

The **order of a DE** is a **n** umber equal to the order of the **h** ighest **d** erivative that appears in the equation.

Example 2. *Determine the order of the following DEs:*

a) $x^{(5)}(t) + x'''(t) + 5x(t) = 0$ 5 is the order.

b) $\frac{d}{dx}\left(y\frac{dy}{dx}\right) = x^2 + 1$ → I could rewrite this as $\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + y\frac{d^2y}{dx^2} = x^2 + 1$

So, the order is 2.

c) $Dy = \sec^2(t)$ The order is 1.

✍ Part c) in the above example uses operator notation to denote the derivative. The D you see here is **n** ot a **v** ariable , but indicates that the dependent variable y is **d** ifferentiated once. In fact, the notation $\frac{1}{D}y$ is sometimes even used to indicate that y is to be integrated. Operator notation is very common in many disciplines; we will talk about this notation in more depth later in the course!

1.4 Linear DEs

These are DEs in which the **d** ependent **v** ariable appears only "linearly." But what does that mean?

Assume that the dependent variable is y. For a DE to be linear, this means that:

- $y, y', y'', \ldots$ never appear multiplied together or divided by one another in a single term, nor can they be raised to any power other than 1 ;
- terms including y and its derivatives may only be multiplied by constants or functions of the *independent* variable and added or subtracted from one another.
- Equations including forms like $\sin(y)$ or $\cos(y')$ or e^y (or just about any function of y or its derivatives!) are not linear!

✍ Note that *any* forms of the *independent variable* may be found in a linear ODE! The restrictions only lie on the dependent variable and its derivatives.

Example 3. *Determine whether the following DEs are linear or nonlinear. If they are nonlinear, explain why.*

a) $y'' + \frac{1}{t-1}y' + \sin(t)y = 0$ Linear!

b) $(x')^3 - (x')^2 = tx$ Nonlinear! (x' is raised to powers other than 1.)

c) $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = e^x$ Nonlinear, due to the "$y \frac{dy}{dx}$" term

d) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = e^{x^2+2x+1} + \sin(\cos(\tan(\operatorname{arccsch}(x))))$ Linear!

Putting all of our rules together, linear ODEs can always be written in what we will call **s**tandard **f**orm :

1.5 Standard form for Linear DEs

$$y^{(n)}(t) + f_{n-1}(t)y^{(n-1)} + \ldots + f_2(t)y''(t) + f_1(t)y'(t) + f_0(t)y(t) = g(t)$$

where $f_0, f_1, \ldots, f_{n-1}$ and g are all functions of the independent variable t.

If the "f" functions happen to all just be **c**onstants , the DE is called a linear ODE with **c**onstant **c**oefficients . In this case, if the independent variable represents time, such DEs are often called **t**ime **i**nvariant , because the left-hand side of the equation would not change at all as time increases.

1.6 Homogeneous DEs

The function $g(t)$ simply represents the collection of any terms appearing in the DE which do not contain the dependent variable, which are always written on the right-hand side by convention. If there are no such terms, the DE is said to be **h**omogeneous. If there are, the DE is **n**onhomogeneous; in this case, $g(t)$ is often called the "**f**orcing **t**erm" for a DE since it will often represent a force acting upon a system.

Example 4. *Determine if the equation is linear. If it is linear, put the DE into standard form and identify whether it is homogeneous or nonhomogeneous.*

$$(x-1)y'' = \frac{1}{4}x^2y' + y + 2$$

Linear!

$$(x-1)y'' - \frac{1}{4}x^2y' - y = 2$$

$$y'' - \frac{1}{4}\frac{x^2}{x-1}y' - \frac{1}{x-1}y = \frac{2}{x-1}$$

← Standard Form.

This DE is nonhomogeneous due to the non-zero right-hand side.