3 Applications of First-Order ODEs

Now that we can identify and solve many different types of first-order ODEs, we can use them to model various situations.

3.1 Orthogonal Trajectories

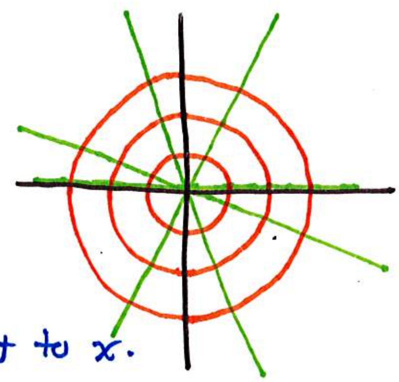
Two intersecting curves are **orthogonal** if their tangent lines are **p**erpendicular at the point of intersection. This can be applied to a number of real-life scenarios:

- In electrostatics: Equipotential surfaces connecting points with the same electrical potential are orthogonal to the lines of an electric field.
- In meteorology: Weather maps feature isobars along which the air pressure is constant. Orthogonal trajectories to these lines connect areas of high pressure to low pressure, giving wind direction.

Remember that, given a slope m, the perpendicular slope is the

n egative **r** eciprocal, given by $\frac{-1}{m}$. So to solve these problems, isolate for the **d** erivative (which represents slope), find the **n** egative **r** eciprocal, and then solve the resulting DE!

Example 1. *Find the orthogonal trajectories to the family of curves given by*



$$x^2 + y^2 = C,$$

where C represents any constant.

First, find $\frac{dy}{dx}$. Take derivative (implicitly) with respect to x.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

So, for orthogonal trajectories, we must solve: $\frac{dy}{dx} = \frac{y}{x}$ (taking the negative reciprocal)

Separable! $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$

Integrate: $\ln|y| = \ln|x| + C$

Isolate for y: $e^{\ln|y|} = e^{\ln|x| + C}$

$\rightarrow \boxed{y = cx}$

Straight lines that pass through the origin!

There is one solution that isn't covered: $x = 0$. This is because $\frac{dy}{dx}$ isn't defined for vertical lines.

But, if we solve the DE: $\frac{dx}{dy} = \frac{x}{y}$ we get the last line!

3.2 Exponential Growth and Decay

Some simple DEs arise very naturally from observing that the **r**ate of **c**hange of particular quantities over time may be directly proportional to the **a**mount of the quantity itself. That is, supposing that $y(t)$ is a quantity, we have that:

$$y'(t) \propto y(t),$$

or as an equation:

$$y'(t) = ky(t),$$

where k is a **p**roportionality constant.

This is a first-order linear (and separable) DE and easily solvable using a variety of methods. Solving, we get:

$$y' = ky$$

$$\frac{1}{y} y' = k$$

$$\ln|y| = kt + C$$

$$\boxed{y = ce^{kt}}$$

Thus, this DE is used to model the case of exponential **g**rowth (if $k > 0$) or **d**ecay (if $k < 0$), and is often used for simple population models, for anything that dissipates over time (heat in an object, one-directional motion with friction involved, etc), and much more.

Example 2. *The growth rate of a population of bacteria is found to be directly proportional to its size. Suppose the population of bacteria is estimated to be* 100,000 *cells at an initial time and* 1,200,000 *cells after 24 hours' time.*

a) Find a function $b(t)$ *that models the growth of the population over time*

$$b' = kb$$

$$\to b = ce^{kt}$$

We have conditions:

$b(0) = 100\,000$

$b(24) = 1\,200\,000$

Applying the first condition,

$$100\,000 = ce^{k(0)} \to \boxed{100\,000 = c}$$

Also, we have:

$$1\,200\,000 = 100\,000e^{k(24)}$$

$$12 = e^{24k}$$

$$\to \ln(12) = 24k$$

$$\to \boxed{k = \frac{1}{24}\ln(12)}$$

So, our function is

$$\boxed{b(t) = 100\,000e^{\frac{1}{24}\ln(12)t}}$$

b) use your function to estimate the size of the population after another 24 hours. After another 24 hours, we are at $t=48$.

We want $b(48)$:

$$b(48) = 100\,000\, e^{\left(\frac{1}{24}\ln(12)\right)48}$$

$$= 100\,000\, e^{2\ln(12)}$$

$$\boxed{= 14{,}400{,}000}$$

✍ Such DEs are usually insufficient for modeling population growth past an initial phase. This makes sense: Populations can only grow within the constraints of their environments, and are subject to challenges such as competition for resources. The basic DE model can be modified to take things like these into consideration.

✍ We can construct other simple DEs that model different situations by using other proportionalities as well. For example, if a known quantity $y(t)$ has a rate of change that is proportional to the amount of the quantity squared — that is, if $y' \propto y^2$ — we obtain a DE of $y' = ky^2$.

3.3 Mixing Problems

Mixing problems usually involve a mass of substance $M(t)$, dissolved in a solution, which is in a tank.

- Solution typically flows into the tank at a particular concentration, **i** nfluencing the concentration of what is already in the tank;
- At the same time, solution may leave the tank at that **m** ixed concentration.
- An **i** nitial **c** ondition may be given or may be calculated using given information.

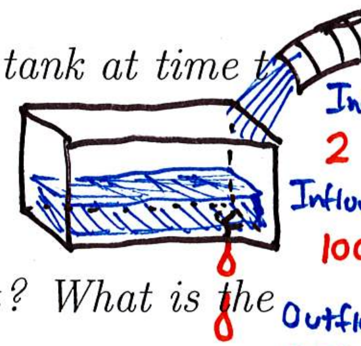
We can construct a DE using the following intuitive rule: The rate of change of the mass of substance **w** ithin the tank is equal to the rate that the mass of substance **e** nters the tank *minus* the rate at which it **e** xits. That is, the DE will look like:

$$\frac{dM}{dt} = \text{Rate in} - \text{Rate out}$$

(of change of mass) (of change of mass)

Units for each term = $\frac{\text{mass}}{\text{time}}$ often

Example 3. *A* 300 *L vat initially contains* 100 *L of vinegar to be made into* 300 *L of brine by adding salt. A* 100 *g/L salt solution is added to the tank at a rate of* 2 *L/sec, but there is an unknown leak in the bottom of the tank; liquid escapes at* 10 *mL/sec. (Assume that the liquids instantly mix uniformly in the vat, and let the mass of salt within the tank at time* t *seconds be given by* $s(t)$*.)*



Inflow Rate: 2 L/s

Inflow Concentration: 100 g/L

Outflow Rate: 0.01 L/s

Outflow Concentration: ??? The concentration of salt in the tank!

a) How much liquid is in the tank at time t*? What is the concentration of salt in the tank at time* t*?*

Amount of Liquid $= 100 + 2t - 0.01t$

So, volume $\boxed{= 100 + 1.99t}$ (Litres)

$$\text{Concentration of salt in tank} = \frac{\text{mass}}{\text{volume}} \boxed{= \frac{s(t)}{100 + 1.99t}} \left(\frac{\text{grams}}{\text{Litre}}\right)$$

b) Construct an IVP for this problem.

$$\text{“}\frac{\text{Mass}}{\text{time}}\text{”} \qquad \frac{ds}{dt} = \text{Rate In} - \text{Rate Out}$$

$$\frac{ds}{dt} = \underbrace{(2\ L/s)}_{\text{Flow Rate In}}\underbrace{(100\ g/L)}_{\text{Conc. In}} - \underbrace{(0.01\ L/s)}_{\text{Flow Rate Out}}\underbrace{\left(\frac{s(t)}{100 + 1.99t}\ g/L\right)}_{\text{Conc. out.}}$$

So, $\boxed{\dfrac{ds}{dt} = 200 - \dfrac{0.01s}{100 + 1.99t}}$

Initial Condition:

No salt initially!

$\boxed{s(0) = 0}$

c) What is the mass of salt in the brine at the instant the vat is full?

First, how much time passes until the vat is full?
Set the expression for volume we found equal to 300 and solve for t:

$$100 + 1.99t = 300$$
$$\rightarrow \frac{200}{1.99} = t$$

Let's solve the IVP:

$$\frac{ds}{dt} + \frac{0.01}{100+1.99t}s = 200$$

Integrating Factor:

$$\mu = e^{\int \frac{0.01}{100+1.99t}dt} = e^{\frac{0.01}{1.99}\int \frac{1.99}{100+1.99t}dt} = e^{\frac{1}{199}\ln|100+1.99t| + C} = e$$

$$\mu = (100+1.99t)^{1/199}$$

Multiply through:

$$(100+1.99t)^{1/199}\frac{ds}{dt} + 0.01(100+1.99t)^{-198/199}s = 200(100+1.99t)^{1/199}$$

Product Rule Backwards:

$$\left((100+1.99t)^{1/199}s\right)' = 200(100+1.99t)^{1/199}$$

Integrate:

$$(100+1.99t)^{1/199}s = \frac{\cancel{200}}{1.99}\frac{(100+1.99t)^{200/199}}{\cancel{200}/199} + C$$

Be very careful with your integration!

$$(100+1.99t)^{1/199}s = 100(100+1.99t)^{200/199} + C$$

Isolate for s:

$$\boxed{s = 100(100+1.99t) + C(100+1.99t)^{-1/199}}$$

General Solution!

Thus, the solution to the IVP is obtained by applying the IC:

$$s(0) = 0 = 100(100) + C(100)^{-1/199} \rightarrow \boxed{C = -10234.114}$$

The solution to the IVP:

$$s(t) = 100(100+1.99t) - 10234.114(100+1.99t)^{-1/199}$$

Finally, to answer the question, find

$$s\left(\frac{200}{1.99}\right) = 100(300) - 10234.114(300)^{-1/199} \boxed{= 20055\,g}$$

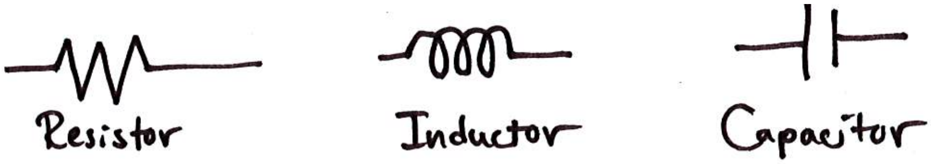
3.4 Simple Electric Circuits

DEs can be used to model an electric circuit that contains resistors, inductors, and/or capacitors. What should the **d**ependent **v**ariable for such a DE be?

With an independent variable t representing time, the dependent variable for the DE could be:

- the charge on the capacitor $q(t)$, measured in **c**oulombs ;
- the flow of electric current $i(t) = \dfrac{dq}{dt}$, in **a**mperes ; or
- the electric potential difference $V(t)$ (in **v**olts.) across a capacitor.

Resistance (R, in **O**hms), inductance (L, in **H**enries), and capacitance (C, in **F**arads) are all positive constants that will appear in our DEs.



From fundamental electricity laws, we have that the voltage drops across these elements in a circuit are given by the following.

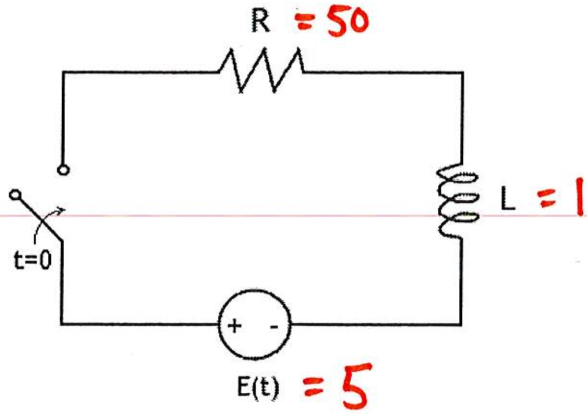
- The voltage drop across a resistor: $V_R = i(t)R$;
- across a capacitor: $V_C = \dfrac{q(t)}{C} = \dfrac{1}{C}\displaystyle\int_{t_0}^{t} i(\tau)\, d\tau$;
- across an inductor: $V_L = L\dfrac{di}{dt}$.

We can also have voltage **r**ises, or sources, which are typically **b**atteries in our circuits. The voltage provided by such a source can be given as a function $E(t)$: For example, $E(t) = 10$ represents a source that provides a constant 10 volts. $E(t) = e^{-50t}$ represents a source where voltage decreases as time goes on.

With all of this, recall **K**irchoff's **v**oltage **l**aw. (KVL), which says: "*The sum of the electrical potential differences (voltage drops or rises) around any loop is zero.*"

This is the rule which we will use to construct all of the circuits-related DEs in this course!

Example 4. *Consider the following circuit diagram:*



Here, the circuit has a resistance of $R = 50\ \Omega$, *an inductance of* $L = 1\ H$, *and a battery that applies a voltage* $E(t)$ *of a constant* $5\ V$. *At time* $t = 0$, *the switch is closed (so at that very instant, assume there is no initial current).*

a) Construct an IVP that models this situation with the current $i(t)$ as the dependent variable.

KVL says that all voltage rises and drops should sum to zero (or equivalently, the sum of the rises should equal the sum of the drops). Here, $V_L + V_R = E(t)$, or subbing in for each:

$$L\frac{di}{dt} + Ri = 5$$

$$\rightarrow (1)\frac{di}{dt} + 50i = 5$$

Our initial condition is given in the problem: $i(0) = \underline{0}$!

b) Solve your IVP.

$$\frac{di}{dt} + 50i = 5$$

Integrating Factor:

$$\mu = e^{\int 50\,dt} = e^{50t}$$

$$e^{50t}\frac{di}{dt} + 50e^{50t}i = 5e^{50t}$$

$$\left(e^{50t}i\right)' = 5e^{50t}$$

$$e^{50t}i = \frac{1}{10}e^{50t} + C$$

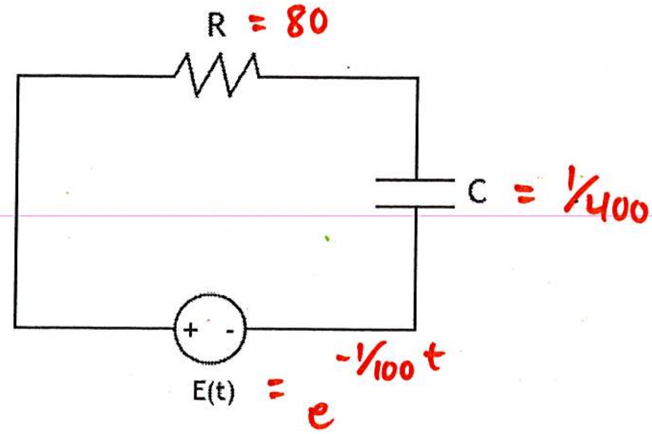
$$i(t) = \frac{1}{10} + Ce^{-50t}$$

Apply IC:

$$i(0) = 0 = \frac{1}{10} + C \rightarrow C = -\frac{1}{10}$$

$$\boxed{i(t) = \frac{1}{10} - \frac{1}{10}e^{-50t}}$$

Example 5. *Consider the following circuit diagram:*



Suppose the battery applies a voltage of $E(t) = e^{-\frac{1}{100}t}$, *the resistance is* $R = 80\ \Omega$, *and the capacitance is* $C = \dfrac{1}{400}$ *F.*

a) Construct an IVP that models this situation with the voltage across the capacitor $V_C(t)$ as the dependent variable, using an arbitrary initial voltage $V_C(0) = V_0$.

$V_R + V_C = E(t)$

Developing a DE in terms of V_C isn't quite as straightforward, but it isn't bad if you remember some calculus tricks. Our voltage drops will be

$$V_R = i(t)R, \qquad \text{and} \qquad V_C = \frac{1}{C}\int_0^t i(\tau)\, d\tau.$$

We use the integral form of V_C because it contains the current $i(t)$, just like V_R does. This allows us to relate the two, so that we can write V_R in terms of V_C.

Taking the derivative of the second equation, we obtain

$$\frac{dV_c}{dt} = \frac{1}{C} i(t)$$

or isolating for current,

$$i(t) = C\frac{dV_c}{dt}$$

This means that $V_R = i(t)R = C\frac{dV_c}{dt}R$! We now have our two voltage drops in terms of V_C and we can use KVL to make our DE:

$$V_R + V_C = E(t)$$

Or subbing in what we know for V_R and $E(t)$,

$$CR\frac{dV_c}{dt} + V_c = E(t)$$

$$\frac{1}{400}(80)\frac{dV_c}{dt} + V_c = e^{-1/100\, t}$$

Our initial condition is given by $V(0) = \underline{V_0}$.

b) Solve your IVP.

$$\frac{1}{5}\frac{dV_c}{dt} + V_c = e^{-1/100\, t}$$

St. Form: $$\frac{dV_c}{dt} + 5V_c = 5e^{-1/100\, t}$$

Int. Factor: $$\mu = e^{\int 5\,dt} = e^{5t}$$

$$e^{5t}\frac{dV_c}{dt} + 5e^{5t}V_c = 5e^{499/100\, t}$$

$$\left(e^{5t}V_c\right)' = 5e^{499/100\, t}$$

$$e^{5t}V_c = \frac{500}{499}e^{499/100\, t} + C$$

$$V_c = \frac{500}{499}e^{-1/100\, t} + Ce^{-5t}$$

Initial Condition:

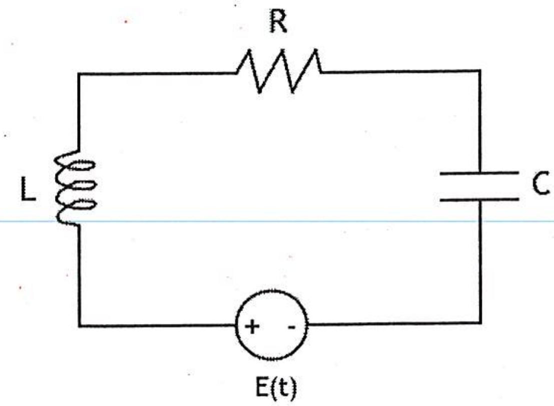
$V(0) = V_0$

$$V_0 = \frac{500}{499} + C \rightarrow \boxed{C = V_0 - \frac{500}{499}}$$

Solution to the IVP:

$$\boxed{V_c(t) = \frac{500}{499}e^{-1/100\, t} + \left(V_0 - \frac{500}{499}\right)e^{-5t}}$$

Example 6. *Consider the following circuit diagram:*



Suppose the applied voltage is $E(t) = 10\sin(\pi t)$, *the inductance is* $L = 0.5$ *H, the resistance is* $R = 40\ \Omega$, *and the capacitance is* $C = \dfrac{1}{1000}$ *F. Construct an IVP, using an arbitrary initial charge* $q(0) = q_0$, *that models this situation with the charge on the capacitor* $q(t)$ *as the dependent variable.*

Putting together a DE in terms of charge is actually very straightforward! Use KVL:

$$V_C + V_R + V_L = E(t)$$

$$q/C + iR + L\frac{di}{dt} = E(t)$$

Note: $i = \dfrac{dq}{dt}$

so $\dfrac{di}{dt} = \dfrac{d^2q}{dt^2}$

$$q/C + \frac{dq}{dt}R + L\frac{d^2q}{dt^2} = E(t)$$

$$0.5\frac{d^2q}{dt^2} + 40\frac{dq}{dt} + 1000q = 10\sin(\pi t).$$

$$q(0) = q_0$$

But this DE is s<u>econd</u> o<u>rder</u>. We don't have an approach for how to solve these... yet. (**T**<u>o</u> **b**<u>e</u> **c**<u>ontinued</u>!)