

MAT 1322 C Winter 2014. Wednesday, March 12 17h30–18h50
Termeh Kousha

Midterm 2, Version A

Last name [CAPITAL]:

First name:

Student Number : _____

- Length: 80 minutes.
- Only those calculators explicitly allowed by the Faculty of Sciences (Texas Instruments TI-30, TI-34 et Casio fx-260, fx-300) are authorized. This is a closed-book, closed-note test.
- Answer each question in the space provided. Use the back of the page for scratch work if necessary. Questions 1 to 4 are multiple choice, each worth 2 points, place your answers to the multiple choice questions in the boxes below. Questions 5 to 7 are long-answer questions. The exam is graded out of 20.

Multiple Choice Answers:

B

#1

C

#2

B

#3

A

#4

Multiple Choice Questions

1. [2 pts] The sum of the series $\sum_{n=2}^{\infty} \frac{4}{n^2-1}$ is

A: $\frac{3}{2}$ B: 3 C: $\frac{9}{2}$ D: 2 E: $\frac{11}{2}$ F: 1

$$\sum_{n=2}^{\infty} \frac{4}{(n-1)(n+1)} = \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n+1} \right) = \left(2 - \frac{2}{3} \right) + \left(1 - \frac{1}{2} \right) + \left(\frac{2}{3} - \frac{2}{5} \right) + \dots + \left(\frac{2}{n-2} - \frac{2}{n} \right) + \left(\frac{2}{n-1} - \frac{2}{n+1} \right) + \dots$$

$$S_n = \sum_{i=2}^n \left(\frac{2}{i-1} - \frac{2}{i+1} \right) = 2 + 1 + \dots - \frac{2}{n} - \frac{2}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = 3$$

2. [2 pts] The sum of the series $3 + 2 + \frac{4}{3} + \frac{8}{9} + \dots$ is

A: $\frac{2}{3}$ B: $\frac{9}{2}$ C: 9 D: 16 E: $\frac{16}{2}$ F: 8

$$a_1 = 3$$

$$ar = 3(r) = 2 \Rightarrow r = \frac{2}{3}$$

$$ar^2 = \frac{4}{3} = 3\left(\frac{2}{3}\right)^2 \quad r < 1$$

⋮

$$S = \frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9$$

3. [2 pts] By using alternating series approach, how many terms at least are required to find the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to accuracy 0.0005?

A: $n = 4$ B: $n = 6$ C: $n = 7$

D: $n = 10$ E: $n = 12$ F: $n = 15$

$$R_n < b_{n+1} < 0.0005 = 5 \times 10^{-4}$$

$$\frac{1}{(n+1)^4} < 5 \times 10^{-4} \Rightarrow (n+1)^4 > 2000$$

$$n+1 > 6.68$$

$$n > 5.68$$

$$\Rightarrow n = 6$$

4. [2 pts] A chocolate donut is taken out of the refrigerator, which has a constant temperature of 4°C and left on the counter. The temperature in the kitchen is constant 20°C . After one hour, the temperature of the donut is 10°C . If the temperature of the donut follows Newton's Law of Heating, what is the temperature after being out of refrigerator for two hours?

A: 13.75°C B: 14.40°C C: 16.67°C

D: 17.54°C E: 19.52°C F: 20.21°C

$T(t)$: temperature of the chocolate donut.

$$T(0) = 4. \quad T(1) = 10$$

Newton's law

$$\frac{dT}{dt} = -K(T - 20) \quad K > 0 \quad (\text{or } \frac{dT}{dt} = K(T - 20) \quad K < 0)$$

$$\frac{dT}{T-20} = -K dt \Rightarrow \ln|T-20| = -Kt + C$$

$$T-20 = Ae^{-Kt} \Rightarrow T = Ae^{-Kt} + 20$$

$$T(0) = 4 \Rightarrow 4 = A + 20 \quad A = -16$$

$$T(t) = -16e^{-Kt} + 20$$

$$T(1) = 10 \Rightarrow 20 - 16e^{-K} = 10 \Rightarrow -16e^{-K} = -10$$

$$T(2) = 20 - 16e^{-0.47(2)} \approx 13.75^\circ\text{C}$$

$$e^{-K} = 5/8 \Rightarrow K = -\ln(5/8) = 0.47$$

Long Answer Questions

5. [6 pts] Determine if the following series are convergent or divergent. Justify your answer.

1 (a) $\sum_{n=1}^{\infty} \frac{5 - \sin(n)}{n}$ $\frac{5 - \sin n}{n} \gg \frac{4}{n} > \frac{1}{n}$ or
 $\sum_{n=1}^{\infty} \frac{4}{n} = 4 \sum_{n=1}^{\infty} \frac{1}{n}$ is Harmonic Series Divergent
 By Comparison Test $\sum \frac{5 - \sin n}{n}$ is also Divergent

2 points (b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$
 $f(x) = \frac{x^2}{e^{x^3}}$ $x \geq 1$ Positive, decreasing [0.5 points] and cont
 $\int_1^{\infty} \frac{x^2}{e^{x^3}} dx = \lim_{d \rightarrow \infty} \int_1^d \frac{x^2}{e^{x^3}} dx = \lim_{d \rightarrow \infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^d = \lim_{d \rightarrow \infty} \left(-\frac{1}{3} e^{-d^3} + \frac{1}{3} e^{-1} \right) = \frac{1}{3e}$
 so By Integral test $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is also convergent
 0.5 points

1 (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3n+1}{7n+5}$
 $\lim_{n \rightarrow \infty} \frac{3n+1}{7n+5} = \frac{3}{7} \neq 0$ and $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} 3n+1}{7n+5} \neq 0$ (DNE)
 Divergent.

2 points (d) $\sum_{n=0}^{\infty} \frac{2n^n}{n!}$
 $\lim_{n \rightarrow \infty} \left| \frac{\frac{2(n+1)^{n+1}}{(n+1)!}}{\frac{2n^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \right|$
 $= \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^n \right| = e > 1$ ← 0.5 point

0.5 point → By Ratio test, **divergent**

and interval
 6. [2 pts] Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n(n+3)}$.

0.5 point

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x+1)^{n+1}}{3^{n+1}(n+4)}}{\frac{(x+1)^n}{3^n(n+3)}} \right| = \lim_{n \rightarrow \infty} |x+1| \left(\frac{1}{3} \frac{n+3}{n+4} \right) = \frac{|x+1|}{3}$$

So the series is convergent : $\frac{|x+1|}{3} < 1 \Rightarrow |x+1| < 3 \Rightarrow R=3$ 0.5 point

$$-3 < x+1 < 3 \Rightarrow -4 < x < 2$$

check the end points

0.5 point $x=2$ $\sum_{n=1}^{\infty} \frac{3^n}{3^n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n+3}$ Divergent (Comparing test to harmonic series $\sum \frac{1}{n}$)

0.5 point $x=-4$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n(n+3)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}$ Convergent By AST

$$\boxed{-4 \leq x < 2}$$

0.5 point

3

7. [4 pts] (a) For all $-1 < x < 1$, find the power series representation for $\ln(4-x)$. (Hint: $\ln(4-x) = -\int \frac{1}{4-x} dx$. Do NOT forget to find the constant of integration.) and its interval of convergence

1 point

$$\frac{1}{4-x} = \frac{1}{4} \left(\frac{1}{1-\frac{x}{4}} \right) = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x}{4} \right)^n = \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}} \quad |x| < 4$$

$$\ln(4-x) = - \int \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}} dx = - \sum_{n=0}^{\infty} \int \frac{x^n}{4^{n+1}} dx = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)4^{n+1}} + C \quad |x| < 4$$

$$x=0 \Rightarrow \ln(4) = C \leftarrow (0.5) \text{ point}$$

1 point

$$(0.5) \text{ point} \rightarrow \ln(4-x) = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)4^{n+1}} + \ln 4 \quad |x| < 4$$

1 points

$$\begin{aligned} x=4 &\Rightarrow - \sum_{n=0}^{\infty} \frac{1}{n+1} \quad D. \\ x=-4 &\Rightarrow - \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)} \quad C \text{ By Ast} \end{aligned} \quad \left. \begin{array}{l} -4 \leq x < 4 \end{array} \right\}$$

1 point

(b) Use part (a) to express $\ln 3$ as the sum of an infinite series.

$$\text{let } x=1 \\ \ln 3 = - \sum_{n=0}^{\infty} \frac{1}{(n+1)4^{n+1}} + \ln 4 =$$

$$\text{or} \quad = \ln 4 - \left(\frac{1}{4} + \frac{1}{2 \cdot 4^2} + \frac{1}{3 \cdot 4^3} + \dots \right)$$

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Midterm 2, Version B

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First name:

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Multiple Choice Answers:

C

#1

D

#2

C

#3

C

#4

Multiple Choice Questions

1. [2 pts] The sum of the series $\sum_{k=2}^{\infty} \frac{6}{k^2-1}$ is

A: $\frac{3}{2}$ B: 3 C: $\frac{9}{2}$ D: 2 E: $\frac{11}{2}$ F: 1

$$\sum_{k=2}^{\infty} \frac{6}{k^2-1} = \sum_{k=2}^{\infty} \frac{3}{k-1} - \frac{3}{k+1} = (3 - 1) + \left(\frac{3}{2} - \frac{3}{4}\right) + \left(1 - \frac{3}{5}\right) + \dots + \left(\frac{3}{k-2} - \frac{3}{k}\right) + \left(\frac{3}{k-1} - \frac{3}{k+1}\right)$$

$$S_n = \sum_{k=2}^n \left(\frac{3}{k-1} - \frac{3}{k+1}\right) = 3 + \frac{3}{2} - \frac{3}{n} - \frac{3}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = 3 + \frac{3}{2} = \frac{9}{2}$$

2. [2 pts] The sum of the series $4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$ is

A: $\frac{2}{3}$ B: $\frac{9}{2}$ C: 9 D: 16 E: $\frac{16}{2}$ F: 8

$$a = 4$$

$$ar = 3 = 4r \Rightarrow r = \frac{3}{4}$$

$$ar^2 = \frac{9}{4} = 4\left(\frac{3}{4}\right)^2$$

⋮

$$S = \frac{4}{1 - \frac{3}{4}} = \frac{4}{\frac{1}{4}} = 16$$

3. [2 pts] By using alternating series approach, how many terms at least are required to find the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ to accuracy 0.002?

- A: $n = 4$ B: $n = 6$ C: $n = 7$
 D: $n = 10$ E: $n = 12$ F: $n = 15$

$$R_n < b_{n+1} < 0.002$$

$$\frac{1}{(n+1)^3} < 0.002 \Rightarrow (n+1)^3 > 500$$

$$n+1 > 7.93$$

$$n > 6.93$$

$$\underline{n=7}$$

4. [2 pts] A chocolate donut is taken out of the refrigerator, which has a constant temperature of 4°C and left on the counter. The temperature in the kitchen is constant 22°C . After one hour, the temperature of the donut is 10°C . If the temperature of the donut follows Newton's Law of Heating, what is the temperature after being out of refrigerator for three hours?

- A: 13.75°C B: 14.40°C C: 16.67°C
 D: 17.54°C E: 19.52°C F: 20.21°C

$$\frac{dT}{dt} = -k(T - 22)$$

so
 (check version A
 for complete
 solution)

$$T(t) = 22 + Ae^{-kt}$$

$$T(0) = 4 \Rightarrow A = -18$$

$$T(1) = 10 = 22 - 18e^{-k}$$

$$\Rightarrow k = 0.405$$

$$T(t) = 22 - 18e^{-0.405t}$$

$$T(3) = 22 - 18e^{-0.405(3)} \approx 16.67^\circ\text{C}$$

Long Answer Questions

5. [6 pts] Determine if the following series are convergent or divergent. Justify your answer.

1 (a) $\sum_{n=1}^{\infty} \frac{3 + \cos(n)}{\sqrt{n}}$ or
 $\frac{3 + \cos(n)}{\sqrt{n}} > \frac{3}{\sqrt{n}} > \frac{1}{\sqrt{n}}$

$\sum \frac{3}{\sqrt{n}}$ or $\sum \frac{1}{\sqrt{n}}$ is Divergent (P-Test $P = +\frac{1}{2} < 1$)
 so By Comparison test $\sum \frac{3 + \cos(n)}{\sqrt{n}}$ also D

(b) $\sum_{n=1}^{\infty} n^3 e^{-n^4}$

2 $f(x) = x^3 e^{-x^4}$ is conti $(1, \infty)$, positive and decreasing
 $\int_1^{\infty} x^3 e^{-x^4} dx = \lim_{d \rightarrow \infty} \int_1^d x^3 e^{-x^4} dx = \lim_{d \rightarrow \infty} \left. -\frac{1}{4} e^{-x^4} \right|_1^d =$

so $\lim_{d \rightarrow \infty} -\frac{1}{4} [e^{-d^4} + \frac{1}{4} e^{-1}] = \frac{1}{4e}$
 By Integral test $\sum n^3 e^{-n^4}$ is also converges

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4n^2 + 1}{9n^2 + 2}$

4

$\lim_{n \rightarrow \infty} \frac{4n^2 + 1}{9n^2 + 2} \neq 0$ $\lim_{n \rightarrow \infty} (-1)^{n+1} \frac{4n^2 + 1}{9n^2 + 2} \neq 0$ DNE
 Divergent.

(d) $\sum_{n=0}^{\infty} \frac{3n^n}{n!}$

2

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{3(n+1)^{n+1}}{(n+1)!}}{\frac{3n^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)(n+1)^n}{(n+1)n!}}{\frac{n^n}{n!}} \right|$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e > 1$$

Divergent
 By ratio test

6. [2 pts] Find the radius of convergence and intervals of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{4^n(n+4)}$.

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x+2)^{n+1}}{4^{n+1}(n+5)}}{\frac{(x+2)^n}{4^n(n+4)}} \right| = \frac{|x+2|}{4} \lim_{n \rightarrow \infty} \frac{n+4}{n+5} = \frac{|x+2|}{4}$$

Convergent when $|x+2| < 4$ $R=4$

$$\begin{aligned} -4 < x+2 < 4 \\ -6 < x < 2 \end{aligned}$$

$$x=2 \Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{4^n(n+4)} = \sum_{n=1}^{\infty} \frac{1}{n+4} \quad \text{Diverges}$$

$$x=-6 \quad \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n(n+4)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+4} \quad \text{Converges By AST}$$

$$\boxed{-6 \leq x < 2}$$

7. [4 pts] (a) For all $-1 < x < 1$, find the power series representation for $\ln(3-x)$. (Hint: $-\ln(3-x) = \int \frac{1}{3-x} dx$. Do NOT forget to find the constant of integration.)

$$\frac{1}{3-x} = \frac{1}{3} \frac{1}{1 - \frac{x}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \quad \left| \frac{x}{3} \right| < 1 \quad |x| < 3$$

$$\ln(3-x) = - \int \frac{1}{3-x} dx = - \int \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} dx$$

$$= - \sum_{n=0}^{\infty} \int \frac{x^n}{3^{n+1}} dx = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)3^{n+1}} + C \quad |x| < 3$$

$$x=0 \Rightarrow \ln 3 = C \Rightarrow \ln(3-x) = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)3^{n+1}} + \ln 3$$

end points

$$x=3 \quad - \sum_{n=1}^{\infty} \frac{3^{n+1}}{(n+1)3^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \quad D$$

$$x=-3 \quad - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = \text{Convergent By AST} \\ \underline{-3 \leq x < 3}$$

(b) Use part (a) to express $\ln 2$ as the sum of an infinite series.

let $x=1$

$$\ln 2 = - \sum_{n=0}^{\infty} \frac{1}{(n+1)3^{n+1}} + \ln 3$$

or

$$\ln 3 - \left(\frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \dots \right)$$