

Solution to Assignment 3

MAT1322D, Fall 2016

1. (2 marks) Find the limit of the sequence $\{a_n, n = 1, 2, 3, \dots\}$, $a_n = \frac{2^n + 3^n}{2^{n+1} + 3^{n+1}}$.

Solution. $\lim_{n \rightarrow \infty} \frac{2^n + 3^n}{2^{n+1} + 3^{n+1}} = \lim_{n \rightarrow \infty} \frac{(2^n/3^n) + 1}{2(2^n/3^n) + 3} = \frac{1}{3}$.

2. (not marked) Consider series $\sum_{n=1}^{\infty} a_n$, where $a_n = \ln\left(1 + \frac{1}{n}\right)$.

- (a) Show that $\lim_{n \rightarrow \infty} a_n = 0$.

Solution. $\lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)\right) = \ln 1 = 0$.

- (b) Find the partial sum $S_k = \sum_{n=1}^k a_n$. Then show that this series diverges.

Solution. Since $1 + \frac{1}{n} = \frac{n+1}{n}$, $\ln\left(1 + \frac{1}{n}\right) = \ln \frac{n+1}{n} = \ln(n+1) - \ln n$.

Then $S_k = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \dots + (\ln(k+1) - \ln k) = \ln(k+1)$.

Since $\lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \ln(k+1) = \infty$, this series diverges.

3. (2 marks) Find the sum of the series $\sum_{n=0}^{\infty} \frac{2^{2n} - 3^n}{5^{n+1}}$.

Solution. $\sum_{n=0}^{\infty} \frac{2^{2n} - 3^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{2^{2n}}{5^{n+1}} - \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}}$. The right-hand side is the difference of two geometric series. The first has first term $a = 1/5$, and common ratio $r_1 = 4/5$, and the second has first term $b = 1/5$ and common ratio $r_2 = 3/5$. The sum of this series is

$$S = \frac{1/5}{1-4/5} - \frac{1/5}{1-3/5} = 1 - \frac{1}{2} = \frac{1}{2}.$$

4. Consider series $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$.

(a) (3 marks) Use integral test to show that this series is convergent. State the conditions why integral test applies.

(b) (not marked) The partial sum $S_{10} = \sum_{n=0}^{10} \frac{1}{1+n^2} \approx 1.9818$. Find an upper bound and a lower bound of the sum of this series. Then find the average of the upper bound and the lower bound to be an estimate of the sum of this series.

Solution. (a) Since $f(x) = \frac{1}{1+x^2}$ is continuous, decreasing and positive, integral test can be used.

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} [\arctan x]_{x=0}^b = \lim_{b \rightarrow \infty} \arctan b = \frac{\pi}{2}.$$

Since the improper integral converges, this series is convergent.

$$(b) \int_a^{\infty} \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_a^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} [\arctan x]_{x=a}^b = \frac{\pi}{2} - \arctan a.$$

$$\text{When } a = 10, \int_{10}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} - \arctan 10 \approx 0.1000.$$

$$\text{When } a = 11, \int_{11}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} - \arctan 11 \approx 0.0907.$$

The sum S of this series is between $1.9818 + 0.0907$ and $1.9818 + 0.1000$, i.e.,

$$2.0725 < S < 2.0818.$$

Taking the middle point of the upper and lower bound to be an estimate of the sum, we have

$$S \approx \frac{2.0725 + 2.0818}{2} = 2.0771. \quad (\text{For your information: A more accurate result is } S = 2.076674\dots. \text{ The error is approximately } 0.0004).$$

5. (3 marks) Determine whether the series $\sum_{n=1}^{\infty} \frac{n + \sin n}{\sqrt{2n^5 - n}}$ converges or diverges by an appropriate test method.

Solution. When $n > 1$, this series is positive, we can use the comparison test.

Since $n + \sin n < 2n$, and $2n^5 - n = n^5 + (n^5 - n) > n^5$, $\frac{n + \sin n}{\sqrt{2n^5 - n}} < \frac{2n}{n^{5/2}} = \frac{2}{n^{3/2}}$. Since the series

$\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ converges, this series $\sum_{n=1}^{\infty} \frac{n + \sin n}{\sqrt{2n^5 - n}}$ converges.

6. (not marked) Determine whether each of the following series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n}{n+1}}$;

(b) $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n}{n^2+1}}$.

Solution. (a) This series diverges because the general term does not approach zero.

(b) This is an alternating series. Since $\sqrt{\frac{n}{n^2+1}}$ decreases and approaches zero when n approaches zero, by the alternating series test, this series converges.

To show that $\sqrt{\frac{n}{n^2+1}}$ is decreasing, let $f(x) = \frac{x}{x^2+1}$. The derivative of this function is

$$f'(x) = -\frac{x^2-1}{(x^2+1)^2}. \text{ When } x > 1, f'(x) < 0. \text{ This function decreases when } x > 1.$$