Solution to Assignment 2

MAT1322D, Fall 2016

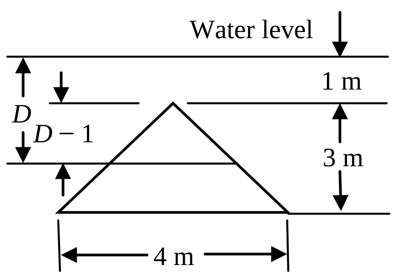
1. (not marked) Construct a definite integral that calculates the length of the arc $y = \sin x$, $0 \le x \le \pi$. Then use the Simpson's method with $n = 6$ to find an approximation of the length of the arc. Use four digits after the decimal point in your calculation.

Solution. $y' = \cos x$. The length of the arc $L = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$. $h = \pi / 6 \approx 0.5236$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$\sqrt{1+\cos^2 x}$	1.4142	1.3229	1.1180	1.0000	1.1189	1.3229	1.4242

$$L = \frac{1}{3} \times 0.5236 \times (1.4142 + 4 \times 1.3229 + 2 \times 1.1180 + 4 \times 1.3229 + 1.4142) \approx 2.7310$$

2. (3 marks) Suppose a triangular surface as shown in the following figure is submerged vertically into water (with density ρ Kg/m^3) so that the top is 1 mater under the water surface:



Let D be the depth of a horizontal slice of the surface. Denote the acceleration of gravity by g. Find the total force acting on this surface, in Newtons. (Use $g = 9.81$ m/sec^2).

Solution. Consider a stripe at depth D with height dD. The area of the stripe is

$$A(D) = \frac{4}{3}(D-1)dD.$$

The pressure is $P(D) = \rho g D$.

The force acting on this stripe is $F(D) = P(D)A(D) = \frac{4}{3}\rho g(D-1)DdD$.

The total force acting on this surface is

$F = \dfrac{4\rho g}{3}\int_1^4 (D-1)D\,dD \approx 1.77 \times 10^5$ Newton.

3. (3 marks) Let R be the region between the graph of $y = \sin x$ and the x-axis, $0 \le x \le \pi/2$. Assuming it has a uniform density $\rho = 1$. Find the moments of R respect to x-axis and y-axis, and coordinates of the center of mass of this region.

Solution. The mass of this plate is $m = \int_0^{\pi/2} \sin x\,dx = 1$.

The moments are $M_x = \dfrac{1}{2}\int_0^{\pi/2} \sin^2 x\,dx = \dfrac{\pi}{8}$, and $M_y = \int_0^{\pi/2} x \sin x\,dx = 1$.

The center of mass is $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(1, \dfrac{\pi}{8}\right)$.

4. (not marked) Use Euler's method with $h = 0.2$ to find an approximation of $y(1)$, where $y(t)$ is the solution of the initial value problem $y' = \sqrt{y} - t$, $y(0) = 1$. Use three digits after the decimal point in your calculation.

Solution. $y_{i+1} = y_i + 0.2(\sqrt{y_i} - t_i)$.

i	t_i	$y_i \approx y(t_i)$
0	0.0	1.0000
1	0.2	1.2000
2	0.4	1.3791
3	0.6	1.5340
4	0.8	1.6617
5	1.0	1.7595

5. (2 marks) Find the solution of the initial-value problem $y' = y(1 + \cos t)$, $y(0) = -1$.

Solution. Separating the variables: $\int \dfrac{1}{y}dy = \int (1+\cos t)dt$.

$\ln|y| = t + \sin t + C$. $|y| = K_1 e^{t + \sin t}$, where $K_1 = e^C > 0$. $y = Ke^{t+\sin t}$, where $K = \pm K_1 \ne 0$.

Using the initial condition, $K = -1$. Hence, the solution to this initial value problem is

$y = -e^{t+\sin t}$.

6. (2 marks) Solve the initial-value problem $y' = 1 - y^2$. $y(0) = 0$.

Solution. This equation has equilibrium solutions $y = 1$ and $y = -1$. When $y \ne \pm 1$,

$$\int \frac{1}{1-y^2}\,dy = \frac{1}{2}\int \left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = t + C\,.$$

$\left|\frac{1+y}{1-y}\right| = K_1 e^{2t}$, where $K_1 = e^{2C} > 0$. $\frac{1+y}{1-y} = K_2 e^{2t}$, where $K_2 = \pm K_1 \neq 0$.

Using the initial condition, $K_2 = 1$. $y = \dfrac{e^{2t}-1}{e^{2t}+1}$.