Solution to Assignment 1

MAT1322D, Fall 2016

1. (not marked) Find the area of the region between the graphs of $y = \sin x$ and $y = \cos x$ in the interval $[0, \pi/2]$.

Solution. The graphs of $y = \sin x$ and $y = \cos x$ has an intersection at $x = \pi/4$. The area of the region is $A =$

$$\int_0^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/3}^{\pi/2}(\sin x - \cos x)dx = \left[\sin x + \cos x\right]_{x=0}^{\pi/4} + \left[-\sin x - \cos x\right]_{x=\pi/4}^{\pi/2} = 2(\sqrt{2} - 1).$$

2. Let R be the region under the graph of $y = \sin x$, $0 \le x \le \pi$.

(a) (not marked) Find the volume of the solid obtained by revolving this region about the line $y = 2$.

(b) (2 marks) Find the volume of the solid by revolving this region about the line $x = -1$ using the method of cylindrical shells.

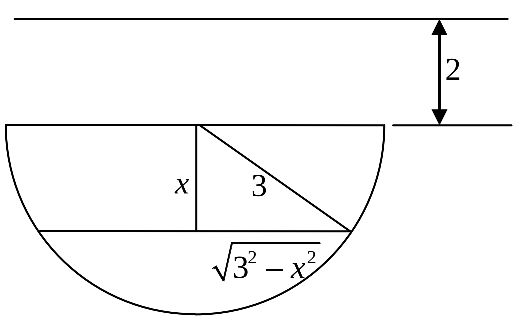
Solution.

(a) $V = \pi\int_0^{\pi}(1-(1-\sin x)^2)dx = \pi\int_0^{\pi}(2\sin x - \sin^2 x)dx = 4\pi - \dfrac{\pi^2}{2}$.

(b) $V = 2\pi\int_0^{\pi}(x+1)\sin x\,dx = 4\pi + 2\pi^2$.

3. (3 marks) A tank is of the shape of the lower half of a sphere with radius 3 meters. It is filled with oil of density $\rho = 850$ kg/m^3. Find the work needed to pump the oil in the tank to a point 2 meters above the top of the tank. Use $g = 9.81$ m^2 / sec.

Solution.



Look at a layer of oil x meters under the top of the tank with thickness dx. The area of the cross section is $A(x) = \pi(9 - x^2)$, and the volume of this layer is $V(x) = A(x)dx = \pi(9 - x^2)dx$. The weight of this layer is $dw = \rho g V(x) = \pi\rho g\,(9 - x^2)dx$. To pump this layer of oil to 2 meters above the top of the tank is $dW = (x + 2)dw = \pi\rho g\,(x + 2)(9 - x^2)dx$. The total work needed is

$$W = \pi\rho g \int_0^3 (x+2)(9-x^2)dx \approx 1.47 \times 10^6 \text{ Joule.}$$

Alternative solutions:

If you let x be the distance between a layer of oil in the tank and the bottom of the tank, then the integral is

$$W = \pi\rho g \int_0^3 (5-x)(9-(3-x)^2)dx\,.$$

If you let x be the distance that a layer of oil to be pumped to 2 meters above the top of the tank, then the integral is

$$W = \pi\rho g \int_2^5 x(9-(x-2)^2)dx\,.$$

Of course, the numerical answers are the same.

4. Use the definition to determine if each of the following improper integral is convergent or divergent. If it is convergent, find its value.

(a) (not marked) $\displaystyle\int_1^2 \frac{1}{\sqrt{2-x}}\,dx$.

(b) (2 marks) $\displaystyle\int_1^\infty \frac{1}{x(x+2)}dx$.

Solution. (a) Use variable substitution $u = 2 - b$.

$$\int_1^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{b\to 2^-}\int_1^b \frac{1}{\sqrt{2-x}}\,dx = -\lim_{b\to 2^-}\int_1^{2-b}\frac{1}{\sqrt{u}}\,du = -2\lim_{b\to 2^-}\left[\sqrt{u}\right]_{u=1}^{2-b} = -2\lim_{b\to 2^-}(\sqrt{2-b}-1) = 2.$$

This improper integral is convergent, and its value is 2.

(b) Use partial fraction.

$$\int_1^\infty \frac{1}{x(x+2)}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x(x+2)}\,dx = \frac{1}{2}\lim_{b\to\infty}\int_1^b\left(\frac{1}{x}-\frac{1}{x+2}\right)dx = \frac{1}{2}\lim_{b\to\infty}\left[\ln x - \ln(x+2)\right]_{x=1}^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\ln\frac{x}{x+2}\right]_{x=1}^{b} = \frac{1}{2}\left(0-\ln\frac{1}{3}\right) = \frac{1}{2}\ln 3.$$

This improper integral is convergent, and its value is $\frac{1}{2}\ln 3$.

5. Use comparison test to determine whether each of the following improper integral is convergent:

(a) (not marked) $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}\,dx$.

(b) (3 marks) $\int_0^1 \frac{2\sqrt{x}+1}{2\sqrt{x}-x}\,dx$.

Solution. (a) Since $2\sqrt{x} - 1 > \sqrt{x}$ and $x + \sqrt{x} < 2x$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} > \frac{\sqrt{x}}{2x} = \frac{1}{2\sqrt{x}}$.

Since $\int_1^\infty \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}\,dx$ diverges.

(b) Since $\sqrt{x} < 1$ when $0 < x < 1$, we have $2\sqrt{x} + 1 < 3$. Since $\sqrt{x} > x$ when $0 < x < 1$, $2\sqrt{x} - x > \sqrt{x}$. Hence, $\frac{2\sqrt{x}+1}{2\sqrt{x}-x} < \frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}}$. Since $\int_0^1 \frac{3}{\sqrt{x}}\,dx = 3\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, $\int_0^1 \frac{2\sqrt{x}+1}{2\sqrt{x}-x}\,dx$ converges.