Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing MAT1348C

Second Midterm Examination — Version α

21 March 2016

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 10 questions on 9 pages. Page 10 contains a list of useful set identities, and may be detached.
- Questions 1-7 are short-answer. Write the final answer in the appropriate answer box, and briefly justify your answer where required.
- Questions 8-10 are long-answer. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be turned off completely and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

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Short-answer questions — write your final answer in the answer box. Wherever indicated, you must briefly justify your answers to receive full marks.

[3pts] 1. In a certain country, a postal code is a string of 5 distinct symbols, with upper case letters and digits alternating. (The strings may start with a letter or a digit.) How many different postal codes are there in this country? *Your answer may contain unevaluated powers, products, and sums.*

Answer: $26 \cdot 10 \cdot 25 \cdot 9 \cdot 24 + 10 \cdot 26 \cdot 9 \cdot 25 \cdot 8$

Justification: LDLDL or DLDLD

$26 \cdot 10 \cdot 25 \cdot 9 \cdot 24 + 10 \cdot 26 \cdot 9 \cdot 25 \cdot 8$

[3pts] 2. How many binary strings of length 7 start with a string 00 **or** end with a string 111? *(This is inclusive or.)* Fully evaluate.

Answer: 44

Justification: $A = \{$strings of len 7 starting with 00$\}$ 00-----

$B = \{$strings of len 7 ending with 111$\}$ ----111

$A \cap B = \{$strings of len 7 starting with 00 and ending with 111$\}$ 00--111

$|A| = 2^5 = 32$

$|B| = 2^4 = 16$

$|A \cap B| = 2^2 = 4$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 32 + 16 - 4 = 44$

3. Let A, B, and C be three subsets of ~~the~~ a non-empty universal set $\mathcal{U}$, and let

$$S = (A - B) \cup \overline{(B \cup C)}.$$

[4pts] Which of the following sets are **equal to** S (for any sets A, B, and C)?

$$\begin{aligned} T_1 &= (A \cup \overline{C}) \cap \overline{B} \\ T_2 &= \overline{\overline{(A \cup \overline{C})} \cup B} \\ T_3 &= \overline{(\overline{A} \cap C) \cup B} \\ T_4 &= \overline{(B - A) \cap \overline{(B \cup C)}} \end{aligned}$$

Answer: T_1, T_2, T_3

No justification is needed.

$S = (A - B) \cup \overline{(B \cup C)} = (A \cap \overline{B}) \cup (\overline{B} \cap \overline{C})$

$= (A \cap \overline{B}) \cup (\overline{C} \cup \overline{B}) = (A \cup \overline{C}) \cap \overline{B} \qquad = T_1$

$= \overline{\overline{(A \cup \overline{C}) \cap \overline{B}}} \quad = \overline{\overline{(A \cup \overline{C})} \cup \overline{\overline{B}}}$

$= \overline{\overline{(A \cup \overline{C})} \cup B} \qquad = T_2$

$= \overline{(\overline{A} \cap C) \cup B} \qquad = T_3$

$T_4 = \overline{(B - A) \cap \overline{(B \cup C)}} = \overline{(B \cap \overline{A})} \cup \overline{\overline{(B \cup C)}}$

$= (\overline{B} \cup A) \cup \overline{(B \cup C)}$

Let $A = B = \mathcal{U}$, $C = \phi$. then $\overline{B \cup C} = \overline{\mathcal{U}} = \phi$.

$T_4 = (\overline{B} \cup A) \cup \phi = \mathcal{U}$

$S = (A - B) \cup \phi = \phi \qquad$ so $S \neq T_4$

[2pts] 4. Let $f : A \to B$ be a bijection. Define the **inverse** of f.

Answer: The inverse of f is a function $f^{-1} : B \to A$ defined by: $f^{-1}(b) = a \iff f(a) = b$

No justification is needed.

5. Let $f : A \to B$ and $g : B \to C$ be two functions. Define the **composition** of these two functions.
[2pts]

Answer: the composition $g \circ f : A \to C$ is defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$

No justification is needed.

with $n \geqslant 2$.

6. Let A and B be finite sets, $|A| = m$ and $|B| = n$, Which of the following statements are **true**?
[5pts]

   (a) If $m = n$, then every function $f : A \to B$ is a bijection. False
   (b) If there exists an injection $f : A \to B$, then $m \leq n$. True
   (c) If there does not exist a surjection $f : A \to B$, then $m < n$. True
   (d) The number of binary relations from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ is $2^{2^{m+n}}$. True
   (e) The number of all functions from A to B is m^n. False

The **true** statements are *(list the letters)*: b, c, d

No justification is needed.

7. Let $\mathcal{U}$ be a non-empty universal set. Define a binary relation $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ as follows:

$$A\mathcal{R}B \quad \leftrightarrow \quad A \cup B = \mathcal{U}.$$

[4pts] Which of the following statements are **true**?

   (a) $\mathcal{R}$ is reflexive. No: $\emptyset \cup \emptyset \neq \mathcal{U}$
   (b) $\mathcal{R}$ is symmetric. Yes: $A \cup B = \mathcal{U} \Rightarrow B \cup A = \mathcal{U}$
   (c) $\mathcal{R}$ is antisymmetric. No: $A \cup \bar{A} = \mathcal{U}$ and $\bar{A} \cup A = \mathcal{U}$, but $A \neq \bar{A}$
   (d) $\mathcal{R}$ is transitive. No: $A \cup \bar{A} = \mathcal{U}$ and $\bar{A} \cup A = \mathcal{U}$, but $A \cup A \neq \mathcal{U}$ if $A \neq \mathcal{U}$

The **true** statements are *(list the letters)*: b

No justification is needed.

Q6 (a) If $B = \{b_1, b_2, .., b_n\}$ and $n \geqslant 2$,

then a function $f: A \to B$, $f(a) = b_1$ for all $a \in A$, is not a bijection.

(b) True, we need $|A| \leq |B|$

(c) Suppose $m \geqslant n$, and $A = \{a_1, a_2, .., a_m\}$, $B = \{b_1, b_2, .., b_n\}$.

Then any function $f: A \to B$ satisfying

$f(a_i) = b_i$ for $i = 1, 2, .., n$, and

and $f(a_i) = b_1$ for $i = n+1, .., m$,

is a surjection.

Hence, if there is no surjection $f: A \to B$, then

$m < n$.

(d) The number of binary relations from a set S to a set T is $|\mathcal{P}(S \times T)|$. Hence the number of bin. relations from $\mathcal{P}(A)$ to $\mathcal{P}(B)$

is $|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| = 2^{|\mathcal{P}(A) \times \mathcal{P}(B)|} =$

$= 2^{|\mathcal{P}(A)|\,|\mathcal{P}(B)|} = 2^{2^{|A|} 2^{|B|}} = 2^{2^m 2^n} = 2^{2^{m+n}}$.

(e) The number of all functions from A to B

is $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m} = n^m$.

Long-answer questions. Detailed solutions are required.

[6pts] 8. For each of the statements below, either prove it or give a concrete, numerical counterexample.

(a) The function $f : \mathbb{Z} \to \mathbb{Z}^2$, defined by $f(x) = (x^2, 3x^2 - 1)$, is **injective**. False

Counter example:

$f(1) = (1, 2) = f(-1)$

(b) The function $g : \mathbb{Z}^2 \to \mathbb{Z}$, defined by $g(x, y) = 2xy$, is **surjective**. False

Counter example:

there is no $(x, y) \in \mathbb{Z}^2$ s.t. $g(x, y) = 1$

(c) Let $A = \{n \in \mathbb{Z} : n \leq 1\}$ and $B = \{n \in \mathbb{Z} : n \geq 3\}$. The function $h : A \to B$, defined by $h(n) = 4 - n$, is a **bijection**. True

To show h is injective:

Take any $m, n \in A$

$h(m) = h(n) \to 4 - m = 4 - n \to m = n$

To show h is surjective:

Take any $b \in B$. We want $a \in A$ s.t.

$h(a) = b$, i.e $4 - a = b$, i.e. $a = 4 - b$.

Since $b \in \mathbb{Z}$ and $b \geq 3$, we have $a \in \mathbb{Z}$

and $a \leq 1$. Thus $a \in A$.

9. Let $\mathcal{R}$ be a binary relation on $\mathbb{R}$ defined by

$$x\mathcal{R}y \quad \leftrightarrow \quad x - y = q \text{ for some rational number } q.$$

[6pts] (a) Prove that $\mathcal{R}$ is an **equivalence relation**. *Fully justify each step.*

$\mathcal{R}$ is reflexive:

For any $x \in \mathbb{R}$, $x - x = 0$ and $0 \in \mathbb{Q}$. Hence $x\mathcal{R}x$.

$\mathcal{R}$ is symmetric:

For any $x, y \in \mathbb{R}$:

$x\mathcal{R}y \rightarrow x - y = q$ for some $q \in \mathbb{Q}$

$\rightarrow y - x = -q$ and $-q \in \mathbb{Q}$

$\rightarrow y\mathcal{R}x$

$\mathcal{R}$ is transitive:

For any $x, y, z \in \mathbb{R}$:

$(x\mathcal{R}y \wedge y\mathcal{R}z) \rightarrow (x - y = q$ and $y - z = r$ for some $q, r \in \mathbb{Q})$

$\rightarrow (x - z = q + r$ and $q + r \in \mathbb{Q})$

$\rightarrow x\mathcal{R}z$

(b) Which of the following sets is the **equivalence class** of 0 with respect to $\mathcal{R}$?

Circle the correct answer:

$\mathbb{R}$ $\quad \{2k+1 : k \in \mathbb{Z}\}$ $\quad \mathbb{Z}$ $\quad \{0\}$ $\quad \emptyset$ $\quad \mathbb{Q}$ $\quad \{2k : k \in \mathbb{Z}\}$

$[0]_{\mathcal{R}} = \{x \in \mathbb{R} : x - 0 \in \mathbb{Q}\} = \mathbb{Q}$

10. Give a **proof by cases** of the following theorem.
[5pts]

Theorem: Let n be an integer. Then $n^2 + 3n + 8$ is even.

Proof. Case 1: n is even.

Then $n = 2k$ for some $k \in \mathbb{Z}$, and

$$n^2 + 3n + 8 = (2k)^2 + 3(2k) + 8 =$$
$$= 4k^2 + 6k + 8 =$$
$$= 2(2k^2 + 3k + 4) = 2m$$

for $m = 2k^2 + 3k + 4$. Since $m \in \mathbb{Z}$ and $n^2 + 3n + 8 = 2m$, $n^2 + 3n + 8$ is even.

Case 2: n is odd

Then $n = 2k+1$ for some $k \in \mathbb{Z}$, and

$$n^2 + 3n + 8 = (2k+1)^2 + 3(2k+1) + 8 =$$
$$= 4k^2 + 4k + 1 + 6k + 3 + 8$$
$$= 4k^2 + 10k + 12$$
$$= 2(2k^2 + 5k + 6) = 2m$$

for $m = 2k^2 + 5k + 6$. Since $m \in \mathbb{Z}$ and $n^2 + 3n + 8 = 2m$, $n^2 + 3n + 8$ is even.

$\square$