Solution to Final Exam

MAT1300, Fall 2013

1. Calculate $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}$.

A) 1; B) 2; C) 3; D) 4; E) 5.

Solution. (C) $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}=\lim_{x\to 2}\dfrac{(x-2)(x+1)}{x-2}=3$.

2. For what value of x does $f(x) = x \ln x$ have slope equal to 0?

A) e; B) e^{-1} C) -1; D) 0; E) None.

Solution. (B) Let $f'(x) = \ln x + 1 = 0$. $\ln x = -1$, $x = e^{-1}$.

3. Use implicit differentiation to find $\dfrac{dy}{dx}$ at (1, 1) when $x^3 + 2x^2y - y^3 + 3y - 5 = 0$.

A) $\dfrac{5}{3}$; B) $\dfrac{3}{2}$; C) $-\dfrac{1}{2}$; D) $\dfrac{2}{5}$; E) $-\dfrac{7}{2}$.

Solution. $3x^2 + 4xy + 2x^2y' - 3y^2y' + 3y' = 0$. At (1, 1), $3 + 4 + 2y' - 3y' + 3y' = 0$. $y' = -\dfrac{7}{2}$.

4. Consider the function $f(x) = x^3 - 3x^2 + 1$. Which of the following statements is correct?

A) There is an inflection point at $x = 1$.
B) There is a local max at $x = 1$.
C) There is a local min at $x = 1$.
D) There is a local max at $x = 4$.
E) There is a local max at $x = 4$.

Solution. (A) $f'(x) = 3x^2 - 6x$. It has critical points $x = 0, 2$. $f''(x) = 6x - 6$. Let $f''(x) = 0$. $x = 1$. The graph is concave up when $x > 1$, and concave down when $x < 1$. $x = 1$ is an inflection point.\

5. Consider the function $g(x) = xe^{-4x}$. Over what interval is the function increasing?

A) $\left(-\infty,\dfrac{1}{4}\right)$; B) $\left(\dfrac{1}{4},\infty\right)$; C) (0, 4); D) $\left(1,\dfrac{5}{4}\right)$; E) $\left(-1,\dfrac{5}{4}\right)$.

Solution. (A) $g'(x) = e^{-4x} - 4xe^{-4x} = e^{-4x}(1 - 4x)$. Critical point: $x = \dfrac{1}{4}$. When $x < \dfrac{1}{4}$, $g'(x) > 0$.

Question 6- Evaluate $\int_0^\infty \frac{x}{(1+x^2)^2}dx =$

(A) 1; (B) 1 / 2; (C) 2; (D) 2 ln 2; (E) The integral is divergent.

Solution. (B) By variable substitution $u = 1 + x^2$.

$$\int_0^\infty \frac{x}{(1+x^2)^2}dx = \lim_{b\to\infty}\int_0^b \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\lim_{b\to\infty}\int_1^{1+b^2}\frac{1}{u^2}du = \frac{1}{2}\lim_{b\to\infty}\left[-\frac{1}{u}\right]_{u=1}^{1+b^2} = \frac{1}{2}\lim_{b\to\infty}\left(1-\frac{1}{1+b^2}\right) = \frac{1}{2}$$

7. Suppose $f'(x) = 4x^3 - 3x^2 + 4x - 3$ and $f(0) = -6$. Find $f(2)$.

A) 0; B) 1; C) 2; D) 3; E) 4.

Solution. (E) $f(x) = x^4 - x^3 + 2x^2 - 3x + C$. $C = -6$. $f(2) = 4$.

8. Suppose that for a certain product, the demand function is given by $D(x) = 20 - 3x^2$ and the supply function is given by $S(x) = x^2 + 4$. Calculate the consumer surplus.

A) 24; B) 16; C) 28; D) 27; E) 13.

Solution. (B) Let $20 - 3x^2 = x^2 + 4$. $x_0 = 2$, $p_0 = 8$. C.S. = $\int_0^2 (20-3x^2-8)dx = 16$.

9. Calculate $\int_1^e x\ln x dx$.

A) e^2; B) $\frac{e^2}{2}$; C) $\frac{e-1}{2}$; D) $\frac{e^2+1}{4}$; E) $\frac{e+3}{2}$.

Solution. (D) $\int_1^e x\ln x dx = \int_1^e \ln x d\left(\frac{1}{2}x^2\right) = \frac{1}{2}\left[x^2\ln x\right]_{x=1}^e - \frac{1}{2}\int_1^e x dx = \frac{e^2}{2} - \frac{1}{4}(e^2-1) = \frac{e^2+1}{4}$.

10. If $f(x, y) = e^{x^2+y^2}$, what is $f_{xy}(1, 1)$?

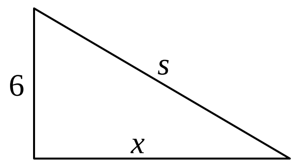
A) e^2; B) $2e^2$; C) $3e^2$; D) $4e^2$; E) $5e^2$.

Solution. $f_x = 2xe^{x^2+y^2}$, $f_{xy} = 4xye^{x^2+y^2}$. $f_{xy}(1, 1) = 4e^2$.

Long Answer Questions

1. (10 points) An airplane flying at a height of 6 miles passes directly over a radar antenna. (See picture.) When the airplane is 10 miles away from the antenna (so $s = 10$), the radar detects that

the distance between the plane and the radar is changing at a rate of 240 miles per hour. What is the speed of the airplane? *HINT*: This question requires related rates.



Solution. Let the horizontal distance between the plane and the antenna be x. Then $x^2 + 6^2 = s^2$. $2xx' = 2ss'$. When $s = 10$, $x = 8$. $s' = 240$, $x' = ss' / x = 300$ miles per hour.

2. (8 points) A manufacturer can produce Red Sox hats at a cost of 9 dollars each. They have been selling the hats at a price of 21 dollars each. At this price they sell 5400 hats per month. For each 1 dollar reduction in price, they sell 300 more hats.

(a) (4 points) Find the demand function. You may assume it is linear.
(b) (4 points) Find the profit function.

Solution. Let te demand function be $p = D(x)$, where $D(x) = mx + b$. $m = -1 / 300$. Since $21 = 5400m + b$, $b = 21 + 18 = 39$. The demand function is $p = -\frac{1}{300}x + 39$. Then the revenue function is $R(x) = -\frac{1}{300}x^2 + 39x$. The Cost function is $C(x) = 9x$. The profit function is $P(x) = R(x) - C(x) = 17x - \frac{1}{300}x^2$.

3. (6 points) Find the absolute maximum for the function $f(x) = x^3 - 3x^2 + 3x + 2$ on the interval $[0, 2]$. Be sure to explain your answer.

Solution. $f'(x) = 3x^2 - 6x + 3$. Since this function is never decreasing in this interval, the absolute maximum is attained at $x = 2$. $f(2) = 4$.

4. (14 points) Consider the two functions: $f(x) = x + 3$ and $f(x) = x^2 + 2x + 1$.

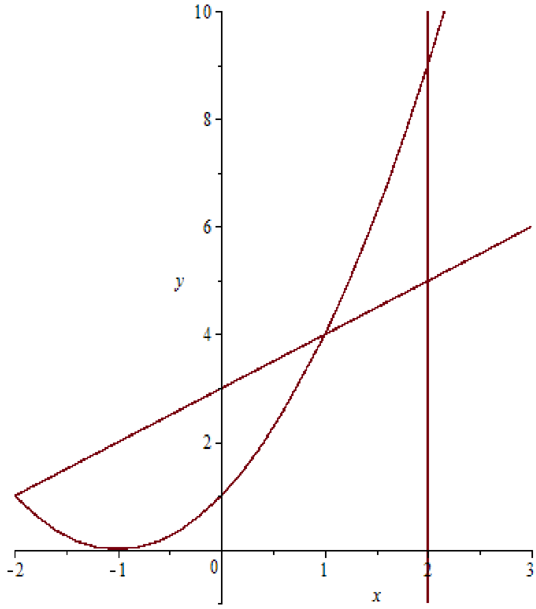
(a) (2 points) Find the intersection points of the graphs of the two functions.

(b) (6 points) On the next page, graph these functions, and shade the region between the graphs of f and g for x such that $0 \le x \le 2$.

(c) (6 points) Find the area of the shaded region.

Solution. Let $x + 3 = x^2 + 2x + 1$. $x^2 + x - 2 = 0$. $x = -2$, $x = 1$. The intersection points are $(-2, 1)$ and $(1, 4)$.

(b)



(c) The area of the region is

$$A = \int_0^1 (x+3-x^2-2x-1)dx + \int_1^2 (x^2+2x+1-x-3)dx = -\int_0^1 (x^2+x-2)dx + \int_1^2 (x^2+x-2)dx$$

Let $F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$. Then $A = F(0) - F(1) + F(2) - F(1) = 3$.

5. (12 points) Consider the function of two variables

$f(x, y) = xy - x^3 + y^3 + 2$.

(a) (3 points) Calculate the first-order partial derivatives.

(b) (3 points) Find all critical points.

(c) (6 points) Identify what type of critical points they are (local max, local min or saddle point).

Solution. (a) $f_x = y - 3x^2$, $f_y = x + 3y^2$.

(b) Let $f_x = 0$. $y = 3x^2$. Let $f_y = 0$. $x = -3y^2$. Then $x = -27x^4$. $x = 0$, or $x = -\frac{1}{3}$.

When $x = 0$, $y = 0$. When $x = -\frac{1}{3}$, $y = \frac{1}{3}$.

Critical points: (0, 0) and $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(c) $f_{xx} = -6x$, $f_{yy} = 6y$, $f_{xy} = 1$. $D = -36xy - 1$.

$D(0, 0) = -1 < 0$. (0, 0) is a saddle point.

$D\left(-\frac{1}{3}, \frac{1}{3}\right) = 3 > 0$. Since $f_{xx}\left(-\frac{1}{3}, \frac{1}{3}\right) = 2 > 0$, $\left(-\frac{1}{3}, \frac{1}{3}\right)$ is a relative minimum.