## Class March 2nd

Thursday, March 2, 2017 11:31 AM

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Audio recording started: 11:31 AM Thursday, March 2, 2017

Ch14: Matrix Multiplication Later today: Ch18 matrix inverses

Last class: • you can multiply two M A & B if
#columns in A = #rows in B

• how to multiply the $(i,j)$ entry of AB is the dot product of row $i$ of A with column $j$ of B

[adjacency matrix] from last class

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \qquad AA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

dot product

these are the number of distinct paths of length from vertex $i$ to vertex $j$

**Properties of matrix multiplication**

• we saw $CD = 0$ when neither C nor D were $0$.

• $A\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$ $B\begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix}$ $C\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ $BC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

→ $AC = BC$ but $A \neq B$

$\boxed{\text{cannot cancel when working with matrices → cannot divide}}$

**Good properties of matrix multiplication**

Theorem: Suppose A, B, C are matrices of the right sizes.
Then, 1) $A(BC) = (AB)C$
associativity

But remember $AB \neq BA$ (typically)
not commutative

• 2) $A(B+C) = AB + AC$
3) $(A+B)C = AC + BC$ *as long as you respect the order

4) If $K \in \mathbb{R}$ then
$K(AB) = (KA)B$
$= A(KB)$

5) Let $O_{nxp}$ be the zero matrix and A is mxn then

$$\underset{m\times n}{A}\ \underset{n\times p}{O} = \underset{m\times p}{O}$$

Similarly $OA = O$
sizes might be different

Example: *pictures from phone*
$(AB)C = A(BC)$

* 2) $u \cdot (v+w) = u \cdot v + u \cdot w$ property of dot product
- "u" rows of A
- "v, w" $j^{th}$ column of B & C

Example: $\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$

$= \begin{bmatrix} a & b & e \end{bmatrix}$

Example: $\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$

Definition: the $n \times n$ identity matrix is called $I$ or $I_n$ and is a diagonal matrix with 1s on the diagonal and 0s elsewhere

eg) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Property: $\underset{m\times n}{A} I_n = A \qquad I_m A = A$

It is an identity I for multiplication

Recall: the transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ where we swapped rows & columns

eg) $A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Theorem: Say $A, B$ are matrices of the right size

1) $(A+B)^T = A^T + B^T$
2) $(kA)^T = kA^T$ for $k \in \mathbb{R}$
3) $(A^T)^T = A$
4) $(\underset{m\times n}{A}\underset{n\times p}{B})^T = B^T A^T$ $\boxed{\text{Swap order!}}$

| check | |
|---|---|
| $A$ $m\times n$ | $B^T$ $p\times n$ |
| $B$ $n\times p$ | $A^T$ $n\times m$ |
| $AB$ $m\times p$ | $B^TA^T$ $p\times m$ |
| $(AB)^T$ $p\times m$ | |

Why are they equal?

$(i, j)$ entry of $AB$ = dot product row $i$ of $A$ & column $j$ of $B$

$(j, i)$ entry $(AB)^T$ = ↑

The $(j,i)$ entry of $B^TA^T$ is dot product of row $j$ of $B^T$ with column $i$ of $A^T$

*picture*

since $u \cdot v = v \cdot u$ (dot product), these answers are the same

eg) $(A+B)(C+D)$
$= (A+B)C + (A+B)D$
$= AC + BC + AD + BD$

eg) $(A+B)(A-B)$
$(A+B)A - (A+B)B$
$= A^2 + \underbrace{BA - AB}_{\neq 0} - B^2$
$\neq A^2 - B^2$

So we can multiply matrices, can we divide?

"NO": ① multiplication is too complicated

② $AC = BC$ but $A \neq B$, so can't divide by $C$

(1)

Perspective: divide by 2 $\Leftrightarrow$ multiply by $\frac{1}{2}$

$$\frac{1}{2} \times 2 = 1$$
$$2 \times \frac{1}{2} = 1$$

**Definition:** Suppose $A$ is a $n \times n$ matrix. If you can find an $n \times n$ matrix $B$ that satisfies $AB = I_n$ and $BA = I_n$

then we say $A$ is <u>invertible</u> and $B$ is the <u>inverse</u> of $A$

We write $B = A^{-1}$ NEVER $\frac{1}{A}$

eg) $A \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ claim: $A$ is invertible & $A$ inverse $= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

<u>check:</u> $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Suppose $A$ is invertible and say $AB = AC$

then $A^{-1}AB = A^{-1}AC$ (multiply both sides on LEFT)

$\Leftrightarrow IB = IC$

$\Leftrightarrow B = C$

Similarly, if $BA = CA$ then $BAA^{-1} = CAA^{-1}$

$\Leftrightarrow BI = CI$

$\Leftrightarrow B = C$

If $AB = CA$

$\Leftrightarrow A^{-1}AB = \underbrace{A^{-1}CA}_{\text{doesn't simplify}}$

but

$\Leftrightarrow B = A^{-1}CA$

In particular, say $\underbrace{A\vec{x} = \vec{b}}_{\text{a system of linear equations}} \xrightarrow{\text{multiply on left by } A \text{ inverse}} \vec{x} = A^{-1}\vec{b}$

multiply by $A$ on the left

Theorem (2x2 matrix inverses)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A$ is invertible if & only if $ad - bc \neq 0$

In which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>check</u> $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left[\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right] = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \overset{=0}{-ab+ba} \\ \underset{=0}{cd-dc} & -cb+da \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

eg) $A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^{-1}\left[\frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\right]$

$= -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Why is $ad - bc \neq 0$ important?

eg) $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\det(C) = 6 - 2 \cdot 3$
$= 0$

say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$

notice $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} d \\ -c \end{bmatrix}}_{\text{vector I just wrote down}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $A^{-1}$ did exist then $A \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow A^{-1} A \begin{bmatrix} d \\ -c \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

<u>But</u> if $A \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ * & * \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

impossible

So if $ad - bc = 0$ then $A^{-1}$ does not exist!

How could we find inverses of $n \times n$ matrices?

eg) $A \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix}$

We want an inverse: a matrix $B$ such that $AB = I_3$

$A \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Leftrightarrow A\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$A\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

This is a collection of linear systems

<u>Recall</u> $\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} x + y + 2z \\ x + 2y + 5z \\ 2x + 2y + 5z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

To solve this, augmented matrix

$\vec{v}_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 5 & 0 \\ 2 & 2 & 5 & 0 \end{array}\right]$

$\vec{v}_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 1 \\ 2 & 2 & 5 & 0 \end{array}\right]$

$\vec{v}_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 2 & 5 & 1 \end{array}\right]$

Solving these systems means doing exactly the same row operations so we do them altogether

$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array}\right]}_{[A \; I_3]} \underset{\substack{-R_1 + R_2 \\ -2R_1 + R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$

consistent, but could not be!

$\underset{\substack{-2R_3 + R_1 \\ -3R_3 + R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 5 & 0 & -2 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$

consistent $\therefore$ invertible

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$$\left[ I_3 \mid A^{-1} \right]$$

<u>check</u>

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & -1 & 1 \\ 1 & 2 & 5 & 5 & 1 & -3 \\ 2 & 2 & 5 & -2 & 0 & 1 \end{array}\right]$$